

Dark Matter Explanation from Quasi-Metric Gravity

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Abstract

Working within the quasi-metric framework, we examine the gravitational field of an isolated, flat disk of spinning dust assuming a weak gravitational field so that relativistic approximations can be used. Contrary to Newtonian theory, these weak field approximations involve a background 3-geometry being that of the 3-sphere rather than Euclidean 3-space.

We solve Laplace’s equation approximately for small distances from the disk and find two discrete spectra of solutions involving Legendre polynomials and Legendre functions, respectively. These spectra correspond to two different mode surface density spectra. The physical surface density of the disk can be expanded in terms of the surface density mode spectrum involving Legendre polynomials by means of a finite Legendre transform. Moreover, boundary conditions single out the exponential disk as a “preferred” physical surface density profile.

On the other hand, the surface density modes involving Legendre functions may also be summed up to yield a second, “induced” surface density (representing collective properties of the disk) that turns out to be singular and thus unphysical. However, it can be used to construct a non-singular “induced associated” surface density and a corresponding “induced associated” potential. The latter contributes to the total potential of the disk and thus influences the rotation curve. However, the nature of the induced associated surface density is geometric rather than material, so it may be perceived as “dark matter”. Applying the derived formulae to an exponential disk, we show that we get an asymptotically flat rotation curve and a correspondence with MOND, in agreement with observations.

1 Introduction

The concept of dark matter (DM), without which mainstream astrophysics and cosmology would not be observationally viable, is an acknowledged part of the modern scientific worldview. Yet all the observational evidence in favour of its existence is based on interpretations of astronomical data coming from distance scales much larger than the solar system. In addition, despite its rather flexible nature, the DM proposal faces

some real observational challenges. In particular galactic phenomenology, including spiral galaxy rotation curve shapes and the seemingly existence of a fixed acceleration scale, seems difficult to understand in terms of DM.

This motivates the alternative approach of trying to explain anomalous galactic observations from modified gravity. The most famous of these approaches is the proposal known as MOND; i.e., a specific modification of Newtonian dynamics whenever the gravitational acceleration falls below a critical value a_0 . Fact is that MOND does a very good job of modelling rotation curve shapes of spiral galaxies just from their visible matter content, using only one freely variable parameter a_0 . That this feat is possible at all, makes the DM approach look suspicious in comparison since no *a priori* reason is given for why a much greater variety of rotation curve shapes is not observed. However, MOND does not work so well on larger distance scales such as clusters of galaxies. Besides, relativistic extensions of MOND involve arbitrary extra fields reducing their predictive powers and thus their advantages vis-a-vis DM.

Rather than trying to modify gravity by introducing extra fields tailored to fit galactic phenomenology, a less contrived approach would be to adopt a general alternative framework of relativistic gravity and see if galactic phenomena can be correctly predicted from first principles. Such an approach can hardly be made to work within the traditional framework of metric theories of gravity [1]. However, another possibility is the so-called quasi-metric framework (QMF) published some time ago [2, 3]. Quasi-metric gravity is rather arcane and it has not yet been shown to be viable. On the other hand, nor is it in obvious conflict with observations, even if it would seem so at first glance. The QMF is also radical inasmuch as the role of the cosmic expansion is described very differently from its counterpart in metric theories of gravity. Specifically, the QMF predicts quantitatively, and from first principles, how the cosmic expansion influences gravitationally bound systems. In particular, the QMF predicts how the cosmic expansion should affect the solar system and that these effects should be observable. A number of observed solar system phenomena have been reinterpreted and shown to be in good agreement with these predictions as long as no extra theory-dependent assumptions are made [4]. Unfortunately, these results indicate that general relativity (GR) is fundamentally flawed and that interpretations of some solar system observations are crucially theory-dependent. Well-known results based on traditional interpretations of such observations may therefore be unreliable. Moreover, questioning traditional interpretations is probably why quasi-metric gravity is ignored by the scientific community, despite making a number of successful predictions not shared with GR [4]. Besides, if it turns out that quasi-metric gravity is able to explain galactic phenomena from first principles as well, this will con-

tribute to undercutting the relevance of the DM concept. In that case the legitimacy of mainstream astrophysics will be weakened even further.

The goal of showing the compatibility of the QMF with galactic phenomena, without the assumption of DM, will be partially fulfilled in this paper. That is, we calculate the gravitational field of a very thin, rotating disk of dust in the non-relativistic limit of quasi-metric gravity. Taking the spinning disk as an idealised model of a spiral galaxy, we indicate a solution of some galactic observations without the need of DM. The correspondence of this solution to that of MOND is also discussed. But first, in the next section, a brief introduction to quasi-metric gravity will be presented.

2 Basic quasi-metric gravity

The QMF has been described in detail elsewhere [2-4]. Here we include only the bare minimum of motivation and general formulae necessary to do the calculations presented in later sections.

The basic motivation for introducing the QMF is the idea that the cosmic expansion should be described as a general phenomenon not depending on the causal structure associated with any semi-Riemannian manifold. And as we will see in what follows, certain properties intrinsic to quasi-metric space-time ensure that this alternative way of describing the cosmic expansion is mathematically consistent and fundamentally different from its counterpart in GR.

Briefly the geometrical basis of the QMF consists of a 5-dimensional differentiable manifold with topology $\mathcal{M} \times \mathbf{R}_1$, where $\mathcal{M} = \mathcal{S} \times \mathbf{R}_2$ is a Lorentzian space-time manifold, \mathbf{R}_1 and \mathbf{R}_2 both denote the real line and \mathcal{S} is a compact 3-dimensional manifold (without boundaries). That is, in addition to the usual time dimension and 3 space dimensions, there is an extra time dimension represented by *the global time function* t . The reason for introducing this extra time dimension is that by definition, t parameterizes any change in the space-time geometry that has to do with the cosmic expansion. By construction, the extra time dimension is degenerate to ensure that such changes will have nothing to do with causality. Mathematically, to fulfil this property, the manifold $\mathcal{M} \times \mathbf{R}_1$ is equipped with two degenerate 5-dimensional (covariant) metrics $\bar{\mathbf{g}}_t$ and \mathbf{g}_t . The metric $\bar{\mathbf{g}}_t$ is found from field equations as a solution, whereas the “physical” metric \mathbf{g}_t can be constructed from $\bar{\mathbf{g}}_t$ in a way described in refs. [2, 3]. (See also section 6.)

The global time function is unique in the sense that it splits quasi-metric space-time into a unique set of 3-dimensional spatial hypersurfaces called *fundamental hypersurfaces* (FHSs). Observers always moving orthogonal to the FHSs are called *fundamental ob-*

servers (FOs). The topology of \mathcal{M} indicates that there also exists a unique “preferred” ordinary global time coordinate x^0 . We use this fact to construct the 4-dimensional quasi-metric space-time manifold \mathcal{N} by slicing the submanifold determined by the equation $x^0 = ct$ out of the 5-dimensional differentiable manifold. (It is essential that this slicing is unique since the two global time coordinates should be physically equivalent; the only reason to separate between them is that they are designed to parameterize fundamentally different physical phenomena.) Thus the 5-dimensional degenerate metric fields $\bar{\mathbf{g}}_t$ and \mathbf{g}_t may be regarded as one-parameter families of Lorentzian 4-metrics on \mathcal{N} . Note that the existence of a “preferred frame” is an intrinsic, *global* geometric property of quasi-metric space-time. (For an isolated system, the preferred frame is the one where the system is at rest, see ref. [5].) Furthermore, there exists a set of particular coordinate systems especially well adapted to the geometrical structure of quasi-metric space-time, *the global time coordinate systems (GTCSs)*. A coordinate system is a GTCS iff the time coordinate x^0 is related to t via $x^0 = ct$ in \mathcal{N} .

For reasons explained in [2, 3], the form of $\bar{\mathbf{g}}_t$ is restricted. Expressed in a GTCS (where the spatial coordinates do not depend on t), the most general form allowed for the family $\bar{\mathbf{g}}_t$ is represented by the family of line elements (this may be taken as a definition)

$$\overline{ds}_t^2 = [\bar{N}^s \bar{N}_s - \bar{N}_t^2](dx^0)^2 + 2\frac{t}{t_0}\bar{N}_i dx^i dx^0 + \frac{t^2}{t_0^2}\bar{N}_t^2 S_{ik} dx^i dx^k. \quad (1)$$

Here t_0 is some arbitrary reference epoch setting the scale of the spatial coordinates, \bar{N}_t is the family of lapse functions of the FOs and $\frac{t_0}{t}\bar{N}^k$ are the components of the shift vector family of the FOs in $(\mathcal{N}, \bar{\mathbf{g}}_t)$. Moreover $S_{ik}dx^i dx^k$ is the metric of \mathbf{S}^3 (with radius equal to ct_0) and $\bar{N}_i \equiv \bar{N}_t^2 S_{ik} \bar{N}^k$. Note that this form of $\bar{\mathbf{g}}_t$ is strictly preserved only under coordinate transformations between GTCSs where the spatial coordinates do not depend on t (but some exceptions to this rule exist). Also note that, since the 5-dimensional metrics are degenerate, there are no components of lapse and shift in the t -direction (i.e., there is no motion and proper time does not elapse along the t -direction).

Due to the restricted form (1) of $\bar{\mathbf{g}}_t$, a full coupling to space-time curvature of the active stress-energy tensor \mathbf{T}_t should not be expected to exist. However, it turns out that a subset of the Einstein field equations can be tailored to $\bar{\mathbf{g}}_t$, so that a *partial* coupling exists [2, 3]. The field equations then read (expressed in a GTCS)

$$\begin{aligned} 2\bar{R}_{(t)\perp\perp} &= 2(c^{-2}\bar{a}_{\mathcal{F}i}^i + c^{-4}\bar{a}_{\mathcal{F}i}\bar{a}_{\mathcal{F}}^i - \bar{K}_{(t)ik}\bar{K}_{(t)}^{ik} + \mathcal{L}_{\bar{\mathbf{n}}_t}\bar{K}_t) \\ &= \kappa(T_{(t)\perp\perp} + \hat{T}_{(t)i}^i), \quad c^{-2}\bar{a}_{\mathcal{F}j} \equiv \frac{\bar{N}_{t,j}}{\bar{N}_t}, \end{aligned} \quad (2)$$

$$\bar{R}_{(t)j\bar{\perp}} = \bar{K}_{(t)j|i}^i - \bar{K}_{t,j} = \kappa T_{(t)j\bar{\perp}}. \quad (3)$$

Here $\bar{\mathbf{R}}_t$ is the Ricci tensor family corresponding to the metric family $\bar{\mathbf{g}}_t$ and the symbol ' $\bar{\perp}$ ' denotes a scalar product with $-\bar{\mathbf{n}}_t$, that is the negative unit normal vector field family of the FHSs. Moreover, $\mathcal{L}_{\bar{\mathbf{n}}_t}$ denotes the Lie derivative in the direction normal to the FHSs, $\bar{\mathbf{K}}_t$ denotes the extrinsic curvature tensor family (with trace \bar{K}_t) of the FHSs, a “hat” denotes an object projected into the FHSs and the symbol ' $|$ ' denotes spatial covariant derivation. Finally $\kappa \equiv 8\pi G/c^4$, where the value of G is by convention chosen as that measured in a (hypothetical) local gravitational experiment in an empty universe at epoch t_0 .

An explicit coordinate expression for $\bar{\mathbf{K}}_t$ may be calculated from equation (1). This expression reads (using a GTCS)

$$\bar{K}_{(t)ij} = \frac{t}{2t_0\bar{N}_t}(\bar{N}_{i|j} + \bar{N}_{j|i}) + \left(\frac{\bar{N}_{t,\bar{\perp}}}{\bar{N}_t} - \frac{t_0}{t}c^{-2}\bar{a}_{\mathcal{F}k}\frac{\bar{N}^k}{\bar{N}_t}\right)\bar{h}_{(t)ij}, \quad (4)$$

$$\bar{K}_t = \frac{t}{t_0}\frac{\bar{N}^i_{|i}}{\bar{N}_t} + 3\left(\frac{\bar{N}_{t,\bar{\perp}}}{\bar{N}_t} - \frac{t_0}{t}c^{-2}\bar{a}_{\mathcal{F}k}\frac{\bar{N}^k}{\bar{N}_t}\right), \quad (5)$$

where $\bar{h}_{(t)ij}$ are the components (in a GTCS) of the spatial metric family $\bar{\mathbf{h}}_t$ intrinsic to the FHSs.

3 Axisymmetric, metrically stationary, flat systems

3.1 Real weak-field solution

The most general form of $\bar{\mathbf{g}}_t$ for an isolated, metrically stationary, axially symmetric system can be found from equation (1). Introducing a spherically symmetric GTCS $\{x^0, \rho, \theta, \phi\}$, where ρ is an isotropic radial coordinate, \bar{N}_t and \bar{N}_ϕ do not depend on ϕ and equation (1) takes the form [5]

$$\bar{ds}_t^2 = \bar{B}\left[-(1 - \bar{V}^2\rho^2\sin^2\theta)(dx^0)^2 + 2\frac{t}{t_0}\bar{V}\rho^2\sin^2\theta d\phi dx^0 + \frac{t^2}{t_0^2}\left(\frac{d\rho^2}{1 - \frac{\rho^2}{\Xi_0^2}} + \rho^2 d\Omega^2\right)\right], \quad (6)$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$, $\Xi_0 \equiv ct_0$, $\bar{B} \equiv \bar{N}_t^2$ and $\bar{V} \equiv \frac{\bar{N}_\phi}{B\rho^2\sin^2\theta}$. The field equations (2), (3) exterior to the source then read [5]

$$\begin{aligned} \left(1 - \frac{\rho^2}{\Xi_0^2}\right)\bar{B}_{,\rho\rho} + \frac{1}{\rho^2}\bar{B}_{,\theta\theta} &+ \frac{2}{\rho}\left(1 - \frac{3\rho^2}{2\Xi_0^2}\right)\bar{B}_{,\rho} + \frac{\cot\theta}{\rho^2}\bar{B}_{,\theta} \\ &= \bar{B}\rho^2\sin^2\theta\left[\left(1 - \frac{\rho^2}{\Xi_0^2}\right)(\bar{V}_{,\rho})^2 + \frac{1}{\rho^2}(\bar{V}_{,\theta})^2\right], \end{aligned} \quad (7)$$

$$(1 - \frac{\rho^2}{\Xi_0^2})\bar{V}_{,\rho\rho} + \frac{1}{\rho^2}\bar{V}_{,\theta\theta} + \left[\frac{4}{\rho} - \frac{5\rho}{\Xi_0^2} + (1 - \frac{\rho^2}{\Xi_0^2})\frac{\bar{B}_{,\rho}}{\bar{B}}\right]\bar{V}_{,\rho} + \left[3\cot\theta + \frac{\bar{B}_{,\theta}}{\bar{B}}\right]\frac{1}{\rho^2}\bar{V}_{,\theta} = 0. \quad (8)$$

For flat systems, it is convenient to switch to a cylindrical GTCS (x^0, ξ, z, ϕ) , where $\xi \equiv \rho \sin\theta$, $z \equiv \rho \cos\theta$. Then equation (6) becomes

$$\begin{aligned} \overline{ds}_t^2 = & \bar{B} \left[- (1 - \bar{V}^2 \xi^2) (dx^0)^2 + 2 \frac{t}{t_0} \bar{V} \xi^2 d\phi dx^0 \right. \\ & \left. + \frac{t^2}{t_0^2} \left((1 - \frac{\xi^2 + z^2}{\Xi_0^2})^{-1} \left[(1 - \frac{z^2}{\Xi_0^2}) d\xi^2 + (1 - \frac{\xi^2}{\Xi_0^2}) dz^2 + 2 \frac{\xi z}{\Xi_0^2} d\xi dz \right] + \xi^2 d\phi^2 \right) \right], \end{aligned} \quad (9)$$

and the field equations (7), (8) read

$$\begin{aligned} (1 - \frac{\xi^2}{\Xi_0^2})\bar{B}_{,\xi\xi} + (1 - \frac{z^2}{\Xi_0^2})\bar{B}_{,zz} - \frac{2\xi z}{\Xi_0^2}\bar{B}_{,\xi z} + \frac{1}{\xi}(1 - \frac{3\xi^2}{\Xi_0^2})\bar{B}_{,\xi} - \frac{3z}{\Xi_0^2}\bar{B}_{,z} \\ = \bar{B}\xi^2 \left[(1 - \frac{\xi^2}{\Xi_0^2})(\bar{V}_{,\xi})^2 + (1 - \frac{z^2}{\Xi_0^2})(\bar{V}_{,z})^2 - \frac{2\xi z}{\Xi_0^2}\bar{V}_{,\xi}\bar{V}_{,z} \right], \end{aligned} \quad (10)$$

$$\begin{aligned} (1 - \frac{\xi^2}{\Xi_0^2})\bar{V}_{,\xi\xi} + (1 - \frac{z^2}{\Xi_0^2})\bar{V}_{,zz} - \frac{2\xi z}{\Xi_0^2}\bar{V}_{,\xi z} + \left[(1 - \frac{\xi^2}{\Xi_0^2})\frac{\bar{B}_{,\xi}}{\bar{B}} - \frac{\xi z}{\Xi_0^2}\frac{\bar{B}_{,z}}{\bar{B}} \right. \\ \left. + \frac{1}{\xi}(3 - \frac{5\xi^2}{\Xi_0^2}) \right]\bar{V}_{,\xi} + \left[(1 - \frac{z^2}{\Xi_0^2})\frac{\bar{B}_{,z}}{\bar{B}} - \frac{\xi z}{\Xi_0^2}\frac{\bar{B}_{,\xi}}{\bar{B}} - \frac{5z}{\Xi_0^2} \right]\bar{V}_{,z} = 0. \end{aligned} \quad (11)$$

We now assume that the gravitational field is weak, so that we may set $\bar{V} = 0$. This implies that equation (11) becomes vacuous and that the right hand side of equation (10) vanishes. To find $\bar{B}(\xi, z)$, we are thus left to solve equation (10) with the right hand side equal to zero. Note that this is equivalent to solving the Laplace equation on a subset of the 3-sphere.

Unfortunately, equation (10) is a non-separable partial differential equation (PDE). This means that there is not much hope of finding exact solutions. However, one may try to find approximate series solutions for $|z| \ll \Xi_0$. The series expansions in terms of $|z|$ should then take the same form as for the corresponding Newtonian problem, recovered by letting $\Xi_0 \rightarrow \infty$ in equation (10). One then gets a separable PDE in the Newtonian potential. That problem was solved many years ago [6, 7], yielding a continuous spectrum of solutions $\Phi_k(\xi, z) \propto J_0(k\xi)\exp(-k|z|)$, where $J_0(k\xi)$ is a Bessel function of the first kind. We are thus led, via correspondence with the Newtonian case, to try solutions $\bar{B}(\xi, z) = 1 + \frac{2}{c^2}\Phi(\xi, z)$ built from mode solutions of the form

$$\Phi_k(\xi, z) = \Phi_k(\xi) \left(1 - k|z| + \frac{1}{2!}\alpha_k(kz)^2 + \dots \right), \quad (12)$$

where the α_k are constants (but higher order coefficients will in general depend on ξ). The reason why the α_k do not depend on ξ , is that any deviation from Euclidean space for small $|z|$ occurs at order $|z|^3$ and higher. That is, by integrating the spatial line element a distance $|z| \ll \Xi_0$ in the z -direction we find (for $\bar{B} \approx 1$)

$$\sqrt{1 - \frac{\xi^2}{\Xi_0^2}} \int_0^z \frac{dz'}{\sqrt{1 - \frac{\xi^2 + z'^2}{\Xi_0^2}}} = z \left(1 + \frac{1}{6(1 - \frac{\xi^2}{\Xi_0^2})} \frac{z^2}{\Xi_0^2} + \dots \right). \quad (13)$$

We now insert equation (12) into equation (10) (with the right hand side set to zero) and collect terms to get separate equations for each power of $|z|$. To lowest order, the terms independent of $|z|$ yield an equation for $\Phi_k(\xi)$, i.e.

$$\left(1 - \frac{\xi^2}{\Xi_0^2}\right) \Phi_{k,\xi\xi} + \frac{1}{\xi} \left(1 - \frac{3\xi^2}{\Xi_0^2}\right) \Phi_{k,\xi} + \alpha_k k^2 \Phi_k = 0. \quad (14)$$

Similarly, collecting terms of first order in $|z|$ yields an equation determining the third order coefficient of the series expansion in equation (12), and so on for each order in $|z|$.

Now, since k is interpreted as a wavenumber on the 3-sphere, it must have a minimum value $k_0 = \frac{1}{\Xi_0}$ corresponding to the maximum value $\xi_{\max} = \Xi_0$. This indicates that, rather than the continuous spectrum of solutions found for the Newtonian case, the solution of equation (14) should involve a *discrete* spectrum of solutions $\Phi_n(\xi) \equiv \Phi_{k_n}(\xi)$. In fact the general solution of equation (14) is (with $\alpha_n \equiv \alpha_{k_n}$)

$$c^{-2} \Phi_n(\xi) = -C_n P_n(u) - C_n^i Q_n(u), \quad n = (\sqrt{\alpha_n k_n^2 \Xi_0^2} + 1)/2, \quad u \equiv 1 - 2 \frac{\xi^2}{\Xi_0^2}, \quad (15)$$

where C_n, C_n^i are (dimensionless) constants and $P_n(u), Q_n(u)$ are Legendre functions of the first and second kind, respectively. (Notice that the mode solutions $C_n^i Q_n(u)$ diverge logarithmically when $u \rightarrow \pm 1$.) From equation (15) we see that we get a discrete spectrum of solutions if n is required to be a non-negative integer. Moreover, we require that k_n should be an integer multiple of k_0 , and that in the continuum limit $\lim_{n \rightarrow \infty} \alpha_n = 1$, so we must choose

$$\alpha_n = 1 - \frac{1}{k_n^2 \Xi_0^2}, \quad \Rightarrow \quad k_n = \frac{2n+1}{\Xi_0}, \quad n = 0, 1, 2, \dots, \quad (16)$$

and this choice will also be consistent with the indicated value of k_0 . Note that, to get the correspondence with the Newtonian case, one may choose $C_n^i = 0$ and $C_n = e^{-s} \frac{s^n}{n!}$ where s is a non-negative real number. Summing over n and using the *generating function*

$$g(u, s) = \exp(su) J_0(s\sqrt{1-u^2}) = \sum_{n=0}^{\infty} \frac{1}{n!} P_n(u) s^n, \quad (17)$$

for Legendre polynomials [8], we get the continuous spectrum of solutions

$$\Phi_s(u) \propto \exp(-s) \sum_{n=0}^{\infty} \frac{1}{n!} P_n(u) s^n = \exp\left(-2s \frac{\xi^2}{\Xi_0^2}\right) J_0\left(2s \frac{\xi}{\Xi_0} \sqrt{1 - \frac{\xi^2}{\Xi_0^2}}\right). \quad (18)$$

Setting $k \equiv \frac{2s}{\Xi_0}$ and then taking the limit $\Xi \rightarrow \infty$ in equation (18), we get back the Newtonian case.

To find the mode surface densities $\bar{\Sigma}_n(u)$ and $\bar{\Sigma}_n^i(u)$ corresponding to the specific mode solutions $-C_n P_n(u)$ and $-C_n^i Q_n(u)$, respectively, we lay a Gauss surface around the disk and use Gauss' theorem across it. This procedure is exactly similar to the Newtonian case treated in [7]. Assuming a weak field ($\bar{B}_n \approx 1$) we find

$$\bar{\Sigma}_n(u) = \frac{c^2 C_n}{2\pi G \Xi_0} (2n+1) P_n(u), \quad \bar{\Sigma}_n^i(u) = \frac{c^2 C_n^i}{2\pi G \Xi_0} (2n+1) Q_n(u). \quad (19)$$

Due to the fact that the set of Legendre polynomials $P_n(u)$, $u \in (-1, 1)$, is complete and orthogonal [9], it is possible to expand any real surface density $\bar{\Sigma}(u)$ in terms of the mode surface densities $\bar{\Sigma}_n(u)$. Setting $C_n \equiv C S_n$ (where C is some nonzero constant) and summing over n , we find

$$\bar{\Sigma}(u) = \sum_{n=0}^{\infty} \bar{\Sigma}_n(u) = \frac{c^2 C}{\pi G \Xi_0} \sum_{n=0}^{\infty} \frac{1}{2} (2n+1) S_n P_n(u). \quad (20)$$

But this means that $\bar{\Sigma}(u)$ is expressed as a *Legendre Fourier series* [9], and that its inverse S_n is the *finite Legendre transform* [9] of $\frac{\pi G \Xi_0}{c^2 C} \bar{\Sigma}(u)$, i.e.

$$S_n = \frac{\pi G \Xi_0}{c^2 C} \int_{-1}^1 \bar{\Sigma}(u') P_n(u') du', \quad (21)$$

so that

$$\bar{\Sigma}(u) = \frac{1}{2} \sum_{n=0}^{\infty} (2n+1) P_n(u) \int_{-1}^1 \bar{\Sigma}(u') P_n(u') du'. \quad (22)$$

The real solution $\Phi(u)$ corresponding to the given surface density $\bar{\Sigma}(u)$ is then obtained by summing up the mode solutions $-C_n P_n(u)$, i.e.

$$\Phi(u) = -c^2 C \sum_{n=0}^{\infty} S_n P_n(u) = -\pi G \Xi_0 \sum_{n=0}^{\infty} P_n(u) \int_{-1}^1 \bar{\Sigma}(u') P_n(u') du'. \quad (23)$$

The corresponding series solution $\bar{B}_{\text{real}}(u, z)$ is then found by combining equations (12), (16), (22) and (23). The result is

$$\begin{aligned} \bar{B}_{\text{real}}(u, z) &= 1 + \frac{2}{c^2} \Phi(u) + \frac{4\pi G}{c^2} |z| \bar{\Sigma}(u) \\ &\quad - \frac{4\pi G}{c^2 \Xi_0} z^2 \sum_{n=0}^{\infty} n(n+1) P_n(u) \int_{-1}^1 \bar{\Sigma}(u') P_n(u') du' + \dots \end{aligned} \quad (24)$$

The circular speed \bar{w}_{real} due to the solution (23) as function of radius of the disk can now be found from the usual non-relativistic formula (this is justified for a weak field $\bar{B}_{\text{real}} \approx 1$, see [4]). Expressed as a function of u we find

$$\begin{aligned}\bar{w}_{\text{real}}^2(u) &= 2\pi G\Xi_0(1-u) \sum_{n=0}^{\infty} \frac{\partial P_n(u)}{\partial u} \int_{-1}^1 \bar{\Sigma}(u') P_n(u') du' \\ &= \frac{2\pi G\Xi_0}{1+u} \sum_{n=0}^{\infty} (n+1) \left(u P_n(u) - P_{n+1}(u) \right) \int_{-1}^1 \bar{\Sigma}(u') P_n(u') du'.\end{aligned}\quad (25)$$

It cannot be expected that rotation curves found from equation (25) should deviate significantly from their Newtonian counterparts (they don't). Thus it would seem that the need for galactic DM is the same as for the standard model. However, here we have not taken into account possible solutions constructed from the specific mode solutions $-C_n^i Q_n(u)$. As we shall see later, if we do this, new possibilities of getting rid of galactic DM open up. But first, in the next section, we need to consider restrictions coming from boundary conditions.

3.2 Boundary conditions

So far we have implicitly assumed that there are no particular preferences regarding the form of $\bar{\Sigma}(u)$ as long as it is physically reasonable and the resulting $\Phi(u)$ is small everywhere. But is this true? To answer that question, we first notice from equation (23) that the real potential at the center of the disk is given by

$$\Phi(1) = -\pi G\Xi_0 \sum_{n=0}^{\infty} \int_{-1}^1 \bar{\Sigma}(u') P_n(u') du' = -\frac{\pi G\Xi_0}{\sqrt{2}} \int_{-1}^1 \frac{\bar{\Sigma}(u') du'}{\sqrt{1-u'}}, \quad (26)$$

where the last expression follows from the generating function [8]

$$\sum_{m=0}^{\infty} P_m(u) s^m = \frac{1}{\sqrt{1-2us+s^2}}, \quad -1 < s < 1, \quad (27)$$

for the borderline case $s = 1$. Moreover, equation (26) may be written in the form

$$\Phi(1) = -c^2 \frac{\bar{\Sigma}_+}{\bar{\Sigma}_*}, \quad \bar{\Sigma}_+ \equiv \frac{1}{2\sqrt{2}} \int_{-1}^1 \frac{\bar{\Sigma}(u') du'}{\sqrt{1-u'}}, \quad \bar{\Sigma}_* \equiv \frac{c^2}{2\pi G\Xi_0}, \quad (28)$$

where $\bar{\Sigma}_+$ is a weighted average, and where the constant $\bar{\Sigma}_*$ sets a specific surface density scale depending on the finite size of space. Besides, a second weighted average surface density $\bar{\Sigma}_-$, related to the total (active) mass M_{t_0} of the disk, can be defined by

$$\bar{\Sigma}_- \equiv \frac{1}{2\sqrt{2}} \int_{-1}^1 \frac{\bar{\Sigma}(u') du'}{\sqrt{1+u'}} = \frac{M_{t_0}}{2\pi\Xi_0^2}, \quad \Rightarrow \quad \Phi(-1) = -c^2 \frac{\bar{\Sigma}_-}{\bar{\Sigma}_*} = -\frac{GM_{t_0}}{\Xi_0}, \quad (29)$$

where $\Phi(-1)$ is found from an expression similar to equation (26) by using equation (27) for the borderline case $s = -1$. We note that $\bar{\Sigma}_*$ is a purely geometric quantity, whereas $\bar{\Sigma}_+$ and $\bar{\Sigma}_-$ depend on the real surface density profile.

Next we note that $\sqrt{|\Phi(-1)|}$ represents a specific (non-vanishing) velocity scale. This indicates the possibility of defining some quantity $\frac{\pi}{\sqrt{2}}[\bar{\sigma}(-1) - \bar{\sigma}(1)] \sim \bar{\Sigma}_*$ with the property that it relates $\Phi(1)$ to $\sqrt{|\Phi(-1)|}$ via a definition similar to equation (28). By combining equations (28) and (29) such a relationship may readily be found. That is, we may define

$$\Phi(1) = -c\sqrt{|\Phi(-1)|} \frac{\bar{\Sigma}_*}{\bar{\Sigma}_-} \frac{\bar{\Sigma}_+}{\bar{\Sigma}_*} \equiv -c\sqrt{|\Phi(-1)|} \frac{\pi}{\sqrt{2}} \frac{[\bar{\sigma}(-1) - \bar{\sigma}(1)]}{\bar{\Sigma}_*}. \quad (30)$$

Notice that equation (30) involves the geometric quantity $\bar{\Sigma}_*$, but since $\Phi(1) = \frac{\bar{\Sigma}_+}{\bar{\Sigma}_-} \Phi(-1)$, the analogous relationship between $\Phi(1)$ and $\Phi(-1)$ does not. Also notice that the factor $\pi/\sqrt{2}$ is included into the definition (30) since $\bar{\sigma}(-1) - \bar{\sigma}(1)$ should be more similar to a mode surface density (see equation (19)) rather than to a weighted average like $\bar{\Sigma}_+$.

A definition similar to equation (30) may be made for the contribution $\Phi_{\geq u}(1)$ to $\Phi(1)$ from the part of the disk interior to some arbitrary coordinate u . The purpose of such a definition is to construct a new “associated” surface density $\bar{\sigma}(u)$. That is, we may define

$$\Phi_{\geq u}(1) \equiv -c\sqrt{|\Phi(-1)|} \frac{\pi}{\sqrt{2}} \frac{[\bar{\sigma}(u) - \bar{\sigma}(1)]}{\bar{\Sigma}_*}, \quad \Phi_{\geq u}(1) \equiv -\frac{\pi G \Xi_0}{\sqrt{2}} \int_u^1 \frac{\bar{\Sigma}(u') du'}{\sqrt{1-u'}}, \quad (31)$$

or equivalently (where the constant $\bar{\sigma}(1)$ must be determined separately, see below)

$$\bar{\sigma}(u) \equiv \bar{\sigma}(1) + \frac{1}{2\pi} \sqrt{\frac{\bar{\Sigma}_*}{\bar{\Sigma}_-}} \int_u^1 \frac{\bar{\Sigma}(u') du'}{\sqrt{1-u'}}, \quad \bar{\sigma}(-1) \equiv \bar{\sigma}(1) + \frac{\sqrt{2}}{\pi} \sqrt{\frac{\bar{\Sigma}_*}{\bar{\Sigma}_-}} \bar{\Sigma}_+. \quad (32)$$

We see from the definition (32) that $\bar{\sigma}(u)$ is increasing from the center of the disk and outwards. Thus, $\bar{\sigma}(u)$ could be interpreted as some kind of “inverted” surface density. This means that $\bar{\sigma}(u)$ should not be considered as an independent, gravitating source. Rather, $\bar{\sigma}(u)$ should give some restrictions on the possible forms of $\bar{\Sigma}(u)$. Such restrictions can be found by requiring that $\bar{\sigma}(u)$ should be linearly related to $\bar{\Sigma}(u)$, so that the corresponding potential can be written as a linear combination of $\Phi(u)$ and some constant potential. Then $\bar{\sigma}(u)$ is not independent. A “preferred” form of $\bar{\sigma}(u)$ can thus be found by requiring that $\bar{\sigma}(u) - \bar{\sigma}(1) + \lambda \bar{\Sigma}(u) = \bar{\sigma}(-1)$, where λ is a constant, or equivalently

$$\bar{\sigma}(u) = \bar{\sigma}(-1) - \frac{\bar{\sigma}(-1)}{\bar{\Sigma}(1)} [\bar{\Sigma}(u) - \bar{\Sigma}(-1)], \quad \Rightarrow \quad \bar{\sigma}(1) = \bar{\Sigma}(-1) \frac{\bar{\sigma}(-1)}{\bar{\Sigma}(1)}. \quad (33)$$

(We see that in this case, since $\bar{\Sigma}(-1)$ should be negligible, $\bar{\sigma}(1)$ should be also.) Equation (33) is then an integral equation determining the most basic form of the real surface

density $\bar{\Sigma}(u)$ for an isolated disk. To find exactly what this form is, it is convenient to turn equation (33) into a first order separable differential equation by taking the derivative w.r.t. u at both sides of it. Solving this equation is straightforward, and the result is an exponential disk, i.e.,

$$\bar{\Sigma}(u) = \bar{\Sigma}(1) \exp \left[- \left(\frac{\bar{\Sigma}(1)}{\bar{\Sigma}_+} - \frac{\bar{\Sigma}(-1)}{\bar{\Sigma}_+} \right) \sqrt{(1-u)/2} \right] \equiv \bar{\Sigma}(1) \exp \left[- \frac{\Xi_0}{\xi_d} \sqrt{(1-u)/2} \right], \quad (34)$$

where $\xi_d \equiv \Xi_0 \bar{\Sigma}_+ / [\bar{\Sigma}(1) - \bar{\Sigma}(-1)]$ is the disk length (at epoch t_0). This result answers the question we posed at the beginning of this section; the simple requirement that $\bar{\Sigma}(u)$ and $\bar{\sigma}(u)$ should be linearly related implies that there is a particular preference regarding the form of $\bar{\Sigma}(u)$. That is, it would seem that the exponential disk should represent a preferred surface density profile among all the possibilities that might exist. This is confirmed observationally, since an exponential surface density is the hallmark density profile of the outer regions of spiral galaxies. We will return to the exponential disk in section 4.

3.3 The induced solution

Contrary to the Legendre polynomials $P_n(u)$, the functions $Q_n(u)$ are not polynomials, and nor do they constitute an orthogonal set for $u \in (-1, 1)$. Rather, the functions $Q_n(u)$ can be separated into two subsets depending on whether n is even or odd. That is, each function with odd n is orthogonal to every function with even n and *vice versa*. On the other hand, functions within each subset are linearly dependent. This can be easily seen from the formulae [8]

$$\begin{aligned} \int_{-1}^1 Q_n(u) Q_n(u) du &= \frac{\frac{\pi^2}{6} + 2 \sum_{k=1}^n \frac{1}{k^2}}{2n+1}, \\ \int_{-1}^1 Q_n(u) Q_m(u) du &= \frac{1 + (-1)^{n-m}}{(m-n)(m+n+1)} \left(\sum_{k=1}^n - \sum_{k=1}^m \right) \frac{1}{k}, \quad n \neq m. \end{aligned} \quad (35)$$

The problem now is to construct a solution $\Phi^i(u)$ from the mode solutions $-C_n^i Q_n(u)$ such that $\Phi(u)$ and $\Phi^i(u)$ are linearly independent, i.e., we require that $\int_{-1}^1 \Phi^i(u) \Phi(u) du = 0$. However, since there are only two sets of linearly independent mode solutions, it must be possible to find many such solutions by summing over different numbers of mode solutions (in general, at least two mode solutions must be included, one from each linearly independent set). This means that, merely requiring linear independence is not sufficient to arrive at a unique solution $\Phi^i(u)$. However, a unique linearly independent solution $\Phi^i(u)$ can indeed be found by summing over *all* the mode solutions. We will call this solution

the induced solution, since it is found indirectly by summing up all the mode solutions $C_n^i Q_n(u)$ such that every term in the mode sum has a linearly independent counterpart $C_n P_n(u)$ from equation (23). Moreover, the requirement of linear independence is not trivial since it forces the constants C_n^i to be dependent on S_n and thus the real surface density. The corresponding surface density $\bar{\Sigma}^i(u)$ will be called *the induced density*. The induced density is not real, but could still have physical consequences indirectly.

The induced solution $\Phi^i(u)$ obtained by summing over all the mode solutions can be found by assuming that the coefficients C_n^i can be written in the form $C_n^i = C^i S_n$, where C^i is a normalisation constant. Then, using the formula [8]

$$\int_{-1}^1 P_n(u) Q_m(u) du = \frac{1 - (-1)^{n+m}}{(n-m)(m+n+1)}, \quad n \neq m, \quad (36)$$

it is easy to see that

$$\int_{-1}^1 \Phi^i(u) \Phi(u) du \propto \sum_{n=0}^{\infty} \sum_{\substack{m=0 \\ m \neq n}}^{\infty} S_n S_m \frac{1 - (-1)^{n+m}}{(n-m)(m+n+1)} = 0, \quad (37)$$

from the obvious antisymmetry obtained by permuting the summation indices. To completely specify the solution $\Phi^i(u)$, it remains to specify the normalisation constant C^i . But the only natural choice is really to set $C^i = C$. We thus have that

$$C_n^i = C_n = C S_n = \frac{\pi G \Xi_0}{c^2} \int_{-1}^1 \bar{\Sigma}(u') P_n(u') du'. \quad (38)$$

The solution $\Phi^i(u)$ is now completely specified, and we have

$$\Phi^i(u) = -c^2 C \sum_{n=0}^{\infty} S_n Q_n(u) = -\pi G \Xi_0 \sum_{n=0}^{\infty} \int_{-1}^1 \bar{\Sigma}(u') P_n(u') du' Q_n(u). \quad (39)$$

Equation (39) may be written in a more convenient form by using the identity [8] (the integral being defined by its Cauchy principal value)

$$Q_n(u) = \frac{1}{2} \int_{-1}^1 \frac{P_n(s) ds}{u - s}, \quad (40)$$

so that by using equation (23) we find that (again using Cauchy principal values)

$$\Phi^i(u) = \frac{1}{2} \int_{-1}^1 \frac{\Phi(s) ds}{u - s}. \quad (41)$$

From equations (19), (22), (38) and (40) we can also find the induced density $\bar{\Sigma}^i(u)$ corresponding to the induced solution $\Phi^i(u)$, i.e.,

$$\bar{\Sigma}^i(u) = \sum_{n=0}^{\infty} \frac{c^2 C_n}{2\pi G \Xi_0} (2n+1) Q_n(u) = \frac{1}{2} \int_{-1}^1 \frac{\bar{\Sigma}(s) ds}{u - s}. \quad (42)$$

However, there is a fundamental problem with the induced quantities. That is, both $\Phi^i(u)$ and $\bar{\Sigma}^i(u)$ will in general contain generic logarithmic divergences (typically located at $u = \pm 1$) inherited from the mode solutions $Q_n(u)$. This means that $\Phi^i(u)$ and $\bar{\Sigma}^i(u)$ cannot be used directly as physical quantities. But $\bar{\Sigma}^i(u)$ may be used *indirectly*, since it is possible to construct a new, geometric, singularity-free surface density from it. We will show how to do this in the next section.

3.4 The induced associated potential

Although $\bar{\Sigma}^i(u)$ cannot be used directly, it still may have physical significance. The reason for this is that it is possible to construct a new surface density $\bar{\sigma}^i(u)$ from $\bar{\Sigma}^i(u)$ using the results found in section 3.2. There, the definition (32) of $\bar{\sigma}(u)$ was motivated from the possibility of restricting possible forms of $\bar{\Sigma}(u)$ due to boundary conditions. However, since $\bar{\sigma}(u)$ is by construction unphysical, there never was any reason to interpret it as an independent gravitating source. On the other hand, there is always the possibility that a definition similar to (32), but with $\bar{\Sigma}^i(u')$ substituted for $\bar{\Sigma}(u')$, may have the properties that makes it possible to use it as a gravitating, geometric source. That is, we may construct the so-called *induced associated surface density* $\bar{\sigma}^i(u)$, defined by

$$\bar{\sigma}^i(u) \equiv \bar{\sigma}^i(1) + \frac{1}{2\pi} \sqrt{\frac{\bar{\Sigma}_*}{\bar{\Sigma}_-}} \int_u^1 \frac{\bar{\Sigma}^i(u') du'}{\sqrt{1-u'}}, \quad \bar{\sigma}^i(1) \approx 0, \quad \frac{\bar{\Sigma}_*}{\bar{\Sigma}_-} = \frac{c^2 \Xi_0}{M_{t_0} G}. \quad (43)$$

Note that, by integrating over $\bar{\Sigma}^i(u')$, its generic logarithmic divergences disappear, so that $\bar{\sigma}^i(u)$ does not contain such singularities. Moreover, since it is possible that $\bar{\Sigma}^i(u)$ may change sign somewhere in the interval $u \in (-1, 1)$, $\bar{\sigma}^i(u)$ may take a form similar to some physical surface density profile. Finally, unlike $\bar{\sigma}(u)$, $\bar{\sigma}^i(u)$ can certainly not be algebraically related to $\bar{\Sigma}(u)$. Thus $\bar{\sigma}^i(u)$ may be considered as a physical, independent gravitating *geometric* quantity, but not as a material density.

We may now use $\bar{\sigma}^i(u)$ as a source in equation (23) to get a new potential, i.e., the so-called *induced associated potential* $\Psi(u)$ defined by

$$\Psi(u) = -\pi G \Xi_0 \sum_{n=0}^{\infty} P_n(u) \int_{-1}^1 \bar{\sigma}^i(u') P_n(u') du'. \quad (44)$$

Since $\bar{\sigma}^i(u)$ does not contain generic divergences, we see from equation (44) that $\Psi(u)$ should be non-singular everywhere, so it may be accepted as a physical quantity. Thus to any real surface density profile $\bar{\Sigma}(u)$ and its corresponding real potential $\Phi(u)$, there is always associated a surface density $\bar{\sigma}^i(u)$ (playing the role of galactic “dark matter”), and

its corresponding potential $\Psi(u)$. These quantities should always be considered together when making predictions.

The series solution $\bar{B}(u, z)$ containing both real and induced associated contributions is then given by

$$\begin{aligned}\bar{B}(u, z) &= 1 + \frac{2}{c^2} \left(\Phi(u) + \Psi(u) \right) + \frac{4\pi G}{c^2} |z| \left(\bar{\Sigma}(u) + \bar{\sigma}^i(u) \right) \\ &- \frac{4\pi G}{c^2 \Xi_0} z^2 \sum_{n=0}^{\infty} n(n+1) P_n(u) \int_{-1}^1 \left(\bar{\Sigma}(u') + \bar{\sigma}^i(u') \right) P_n(u') du' + \dots\end{aligned}\quad (45)$$

Note that the second order term in equation (45) is divergent for $u = 1, z \neq 0$. See the next section for more comments. Moreover, the (non-relativistic) circular speed \bar{w}_{circ} calculated from equation (45) is due to both real matter and induced associated “phantom” matter, and similar to equation (25) we find that

$$\bar{w}_{\text{circ}}^2(u) = \frac{2\pi G \Xi_0}{1+u} \sum_{n=0}^{\infty} (n+1) \left(u P_n(u) - P_{n+1}(u) \right) \int_{-1}^1 \left(\bar{\Sigma}(u') + \bar{\sigma}^i(u') \right) P_n(u') du'. \quad (46)$$

We see from equations (41) and (42) that the induced solutions and densities are obtained by integrating real solutions and densities over the whole disk; thus these quantities describe non-local, *collective* properties of the system. This is also true for $\bar{\sigma}^i(u)$. That is why any extra gravitational acceleration obtained from $\Psi(u)$ should not be interpreted as a fundamental modification of the Newtonian force law such as in MOND - rather the extra acceleration should be seen as an emergent property of the whole system.

3.5 An important transformation

Given as input the real surface density $\bar{\Sigma}(u)$, one should now in principle be able to calculate the real potential $\Phi(u)$, the real series solution $\bar{B}_{\text{real}}(u, z)$, the corresponding induced quantities $\bar{\Sigma}^i(u)$ and $\Phi^i(u)$, the induced associated quantities $\bar{\sigma}^i(u)$ and $\Psi(u)$, the total series solution $\bar{B}(u, z)$, and finally the rotation curve from equation (46). However, as seen from equations (23) and (44), the expressions for $\Phi(u)$ and $\Psi(u)$ contain an infinite sum over (a product of) Legendre polynomials. Since this sum will in general converge slowly, its presence makes numerical calculations quite awkward. Fortunately, it is possible to rewrite this infinite sum in terms of an elliptic integral, making numerical calculations much easier. The key to this important transformation is using the generating function [10] ($|s| < 1$)

$$\sum_{m=0}^{\infty} P_m(u) P_m(u') s^m = \frac{1}{\pi} \int_0^{\pi} \frac{d\omega}{\sqrt{1 - 2s \left(uu' + \sqrt{(1-u^2)(1-u'^2)} \cos \omega \right) + s^2}}, \quad (47)$$

for the borderline case $s = 1$. (Note that, for the special case $u' = 1$, we get back equation (27).)

We now use equation (47) to rewrite equations (23) and (44). Interchanging the sum and the integral in these equations, adding them and then using equation (47), we get

$$\begin{aligned}
\Phi(u) + \Psi(u) &= -\frac{G\Xi_0}{\sqrt{2}} \int_0^\pi \int_{-1}^1 \frac{[\bar{\Sigma}(u') + \bar{\sigma}^i(u')] du' d\omega}{\sqrt{1 - uu' - \sqrt{(1 - u'^2)(1 - u^2)} \cos \omega}} \\
&= -\sqrt{2} G\Xi_0 \int_{-1}^1 \frac{[\bar{\Sigma}(u') + \bar{\sigma}^i(u')] K\left(\sqrt{\frac{2\sqrt{(1 - u^2)(1 - u'^2)}}{1 - uu' + \sqrt{(1 - u^2)(1 - u'^2)}}}\right) du'}{\sqrt{1 - uu' + \sqrt{(1 - u'^2)(1 - u^2)}}} \\
&= -2\sqrt{2} G\Xi_0 \int_{-1}^1 \frac{[\bar{\Sigma}(u') + \bar{\sigma}^i(u')] K\left(\frac{\sqrt{(1 - u^2)(1 - u'^2)}}{1 - uu' + |u - u'|}\right) du'}{\sqrt{1 - uu' + \sqrt{(1 - u'^2)(1 - u^2)}} + \sqrt{1 - uu' - \sqrt{(1 - u'^2)(1 - u^2)}}}, \quad (48)
\end{aligned}$$

where $K(k) \equiv \int_0^{\pi/2} d\theta / \sqrt{1 - k^2 \sin^2 \theta}$ is the complete elliptic integral of the first kind [8]. The last form (see [10]) of equation (48) is a little better to use for numerical purposes.

Similarly, it may also be tempting to interchange the infinite sum and the integral present in the second-order term of equation (45), and then transform the infinite sum into an integral. However, as we shall see, this procedure does not work for higher-order terms, since all will be divergent. To illustrate this, a somewhat lengthy calculation using equation (47) yields

$$\begin{aligned}
\sum_{m=0}^{\infty} m(m+1) P_m(u) P_m(u') &= \lim_{s \rightarrow 1} \left[\frac{\partial}{\partial s} s^2 \frac{\partial}{\partial s} \sum_{m=0}^{\infty} P_m(u) P_m(u') s^m \right] \\
&= \frac{E(k) - (1 - uu' - \sqrt{(1 - u^2)(1 - u'^2)}) K(k)}{2\sqrt{2}\pi |u - u'| \sqrt{1 - uu' - \sqrt{(1 - u'^2)(1 - u^2)}}}, \quad k^2 \equiv \frac{2\sqrt{(1 - u^2)(1 - u'^2)}}{1 - uu' + \sqrt{(1 - u^2)(1 - u'^2)}}, \quad (49)
\end{aligned}$$

where $E(k) \equiv \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} d\theta$ is the complete elliptic integral of the second kind [8]. To get an expression for $\bar{B}(u, z)$ (for small $|z|$) more suitable for numerical calculations, one may now insert equation (48) into equation (45). However, if one tries to insert equation (49) into the quadratic term, interchanging the infinite sum and the integral, it is straightforward to see that this term will diverge. This behaviour is quite similar to the Newtonian case (where the potential takes the form of a double improper integral [7]) if one tries to expand the Newtonian potential in powers of $|z|$; interchanging the two improper integrals implies that all terms with power ≥ 2 will diverge, yet if all the terms are included, the resulting exact result is finite. But only including the linear term in

the series expansion approximates the exact result well for small enough $|z|$. One expects that this is valid for equation (45) also, so that skipping terms with power ≥ 2 is justified for small $|z|$. Note that in equation (45) (also just as for its Newtonian counterpart), all terms of order ≥ 2 will diverge on the z -axis for $z \neq 0$.

Of course, if $|z|$ is not small enough, the second order term (and possibly higher-order terms) must be calculated in some admissible way and included. That is, it would be preferable to find some other way of rewriting the series expansion in $|z|$ (i.e., without using equation (49)), such that all terms can be expressed in a form suitable for numerical calculations. Fortunately, such a form for the second order term can be found from equations (45) and (10) (with $\bar{V} = 0$). From these equations we find that (for small $|z|$)

$$\begin{aligned}\bar{B}_{,zz} &= -\frac{2}{c^2} \left[\left(1 - \frac{\xi^2}{\Xi_0^2}\right) (\Phi_{,\xi\xi} + \Psi_{,\xi\xi}) + \frac{1}{\xi} \left(1 - \frac{3\xi^2}{\Xi_0^2}\right) (\Phi_{,\xi} + \Psi_{,\xi}) \right] + O(|z|) \\ &= -\frac{8}{c^2 \Xi_0^2} \left[(1+u) (\Phi_{,uu} + \Psi_{,uu}) - 2u (\Phi_{,u} + \Psi_{,u}) \right] + O(|z|),\end{aligned}\quad (50)$$

so that the first few terms of the rewritten series expansion read

$$\begin{aligned}\bar{B}(u, z) &= 1 + \frac{2}{c^2} \left(\Phi(u) + \Psi(u) \right) + \frac{4\pi G}{c^2} |z| \left(\bar{\Sigma}(u) + \bar{\sigma}^i(u) \right) \\ &\quad - \frac{4z^2}{c^2 \Xi_0^2} \left[(1+u) (\Phi_{,uu} + \Psi_{,uu}) - 2u (\Phi_{,u} + \Psi_{,u}) \right] + O(|z^3|).\end{aligned}\quad (51)$$

Note that the second order and higher-order terms are still expected to diverge in the limit $u \rightarrow 1$, $z \neq 0$. This means that, to do calculations to a given accuracy at some fixed $z \neq 0$, one needs to include ever more higher-order terms into the series expansion when moving towards the z -axis.

Finally, interchanging the sum and the integral in equation (46), yields, after some tedious calculations, that

$$\begin{aligned}\bar{w}_{\text{circ}}^2(u) &= \sqrt{2} G \Xi_0 \left\{ \frac{u}{1+u} \int_{-1}^1 \frac{\left[\bar{\Sigma}(u') + \bar{\sigma}^i(u') \right] K \left(\sqrt{\frac{2\sqrt{(1-u^2)(1-u'^2)}}{1-uu' + \sqrt{(1-u^2)(1-u'^2)}}} \right) du'}{\sqrt{1-uu' + \sqrt{(1-u'^2)(1-u^2)}}} \right. \\ &\quad \left. - \frac{1}{1+u} \int_{-1}^1 \frac{(u-u') \left[\bar{\Sigma}(u') + \bar{\sigma}^i(u') \right] E \left(\sqrt{\frac{2\sqrt{(1-u^2)(1-u'^2)}}{1-uu' + \sqrt{(1-u^2)(1-u'^2)}}} \right) du'}{|u-u'| \sqrt{1-uu' - \sqrt{(1-u'^2)(1-u^2)}}} \right\}.\end{aligned}\quad (52)$$

Inserting any given surface density $\bar{\Sigma}(u)$ and its induced associated surface density $\bar{\sigma}^i(u)$ into equation (52) now yields the full rotation curve of the disk.

4 The exponential disk

4.1 Approximate solutions

For spiral galaxies, the observed general trend is that surface brightness (and thus the luminosity due to stars in the disk) falls off exponentially from the center and outwards. Therefore, to explain spiral galaxy rotational curves without dark matter, one is required to assume that surface density profiles of stars are proportional to luminosity profiles. This means that, in the general, but idealised case of a truncated disk of extension ξ_0 , we must have that

$$\bar{\Sigma}(\xi) = \begin{cases} \bar{\Sigma}_c \exp(-\xi/\xi_d) & \xi \leq \xi_0, \\ 0 & \xi > \xi_0, \end{cases} \quad \bar{\Sigma}(u) = \begin{cases} \bar{\Sigma}_c \exp\left(-\frac{\Xi_0}{\xi_d} \sqrt{\frac{1-u}{2}}\right) & 1 \geq u \geq u_0, \\ 0 & u < u_0, \end{cases} \quad (53)$$

where $\bar{\Sigma}_c$ is the central surface density and ξ_d is the disk length (both taken at epoch t_0). Note that, since $\xi_d \ll \Xi_0$, $\bar{\Sigma}(\xi)$ falls off so fast that we are justified in treating spiral galaxies as truncated disks for sufficiently large ξ_0 , i.e., for $\xi_0 \gg \xi_d$. For computational purposes, serious errors are hardly made if we choose ξ_0 to lie somewhere in the interval $20 \xi_d \leq \xi_0 \ll \Xi_0$. On the other hand, truncation inevitably yields truncation singularities. However, their contributions to calculations are usually negligible. In what follows, we will assume that the disk is not truncated, but the results are not significantly affected if it is.

Now we may calculate the induced surface density $\bar{\Sigma}^i(\xi)$ associated with the exponential disk. To do that, we compute the integral

$$\begin{aligned} & \int_{-1}^1 (u-s)^{-1} \exp\left(-\frac{\Xi_0}{\xi_d} \sqrt{\frac{1-s}{2}}\right) ds \\ &= \exp\left(\frac{\Xi_0}{\xi_d} \sqrt{\frac{1-u}{2}}\right) \left\{ \text{Ei}\left[-\frac{\Xi_0}{\xi_d} \left(1 + \sqrt{\frac{1-u}{2}}\right)\right] - \text{Ei}\left[-\frac{\Xi_0}{\xi_d} \sqrt{\frac{1-u}{2}}\right] \right\} \\ &+ \exp\left(-\frac{\Xi_0}{\xi_d} \sqrt{\frac{1-u}{2}}\right) \left\{ \text{Ei}\left[-\frac{\Xi_0}{\xi_d} \left(1 - \sqrt{\frac{1-u}{2}}\right)\right] - \text{Ei}\left[\frac{\Xi_0}{\xi_d} \sqrt{\frac{1-u}{2}}\right] \right\} \\ &\approx -\exp\left(\frac{\Xi_0}{\xi_d} \sqrt{\frac{1-u}{2}}\right) \text{Ei}\left[-\frac{\Xi_0}{\xi_d} \sqrt{\frac{1-u}{2}}\right] - \exp\left(-\frac{\Xi_0}{\xi_d} \sqrt{\frac{1-u}{2}}\right) \text{Ei}\left[\frac{\Xi_0}{\xi_d} \sqrt{\frac{1-u}{2}}\right], \end{aligned} \quad (54)$$

where the approximation holds as long as u is not very close to -1 , and if $\xi_d \ll \Xi_0$. Here $\text{Ei}(x)$ is the exponential integral defined by [8]

$$\text{Ei}(x) \equiv -\int_{-x}^{\infty} \frac{\exp(-s)}{s} ds = \int_{-\infty}^x \frac{\exp(s)}{s} ds, \quad \text{Ei}(-x) \equiv -\int_x^{\infty} \frac{\exp(-s)}{s} ds, \quad x > 0. \quad (55)$$

We then find from equation (42) that

$$\bar{\Sigma}^i(\xi) \approx -\frac{\bar{\Sigma}_c}{2} \left\{ \exp(\xi/\xi_d) \text{Ei}(-\xi/\xi_d) + \exp(-\xi/\xi_d) \text{Ei}(\xi/\xi_d) \right\}, \quad 0 < \xi < \Xi_0. \quad (56)$$

Note that for (moderately) large distances, unlike $\bar{\Sigma}(\xi)$, $\bar{\Sigma}^i(\xi)$ tends slowly towards zero (from below) with increasing ξ . Besides, note that $\bar{\Sigma}^i(\xi)$ is positive for small ξ and that it diverges logarithmically towards the origin. (As can be seen from equation (54), the exact expression for $\bar{\Sigma}^i(\xi)$ also has a similar singularity when $\xi \rightarrow \Xi_0$.)

The induced associated surface density $\bar{\sigma}^i(\xi)$ is now straightforwardly found from equations (43) and (56). Neglecting the very small quantity $\bar{\sigma}^i(0)$, we find that (setting $M_{t_0} \approx 2\pi\bar{\Sigma}_c\xi_d^2$, i.e., neglecting corrections coming from the finite size of space)

$$\bar{\sigma}^i(\xi) \approx \frac{c}{2\sqrt{2}\pi^2\xi_d} \sqrt{\frac{M_{t_0}}{G\Xi_0}} \left\{ \exp(-\xi/\xi_d) \text{Ei}(\xi/\xi_d) - \exp(\xi/\xi_d) \text{Ei}(-\xi/\xi_d) \right\}, \quad 0 \leq \xi \leq \Xi_0. \quad (57)$$

Note that $\bar{\sigma}^i(\xi)$ (even using the exact expression for $\bar{\Sigma}^i(\xi)$ found from equation (54)) is non-singular and non-negative everywhere. Thus $\bar{\sigma}^i(\xi)$ may formally play the role of DM.

Next we find an approximative expression (valid if $\xi \ll \Xi_0$) for the real potential $\Phi(\xi)$ from equation (48) (omitting $\bar{\sigma}^i(u)$). Splitting the integral up into two improper parts, we find that

$$\Phi(\xi) \approx -4G\bar{\Sigma}_c\xi_d \left[\int_0^{\frac{\xi}{\xi_d}} + \int_{\frac{\xi}{\xi_d}}^{\frac{\Xi_0}{\xi_d}} \right] \frac{x \exp(-x) K\left(2\frac{\sqrt{x\xi/\xi_d}}{x+\xi/\xi_d}\right) dx}{x + \xi/\xi_d}, \quad \xi \ll \Xi_0. \quad (58)$$

Note that, as can be readily verified numerically, $\Phi(\xi)$ as calculated from equation (58) is essentially identical to its Newtonian counterpart $\Phi_N(\xi)$ (valid for an infinite exponential disk), given by [7]

$$\Phi_N(\xi) = -\pi G\bar{\Sigma}_c\xi \left[I_0(\xi/2\xi_d) K_1(\xi/2\xi_d) - I_1(\xi/2\xi_d) K_0(\xi/2\xi_d) \right], \quad (59)$$

where $I_\nu(x)$ and $K_\nu(x)$ are modified Bessel functions [8]. Similarly, the corresponding circular speed \bar{w}_{real} can be found approximately from equation (52) (omitting $\bar{\sigma}^i(u)$), and this is also numerically very close to its Newtonian counterpart. That is, again splitting up integrals into improper parts, we find that for $\xi \ll \Xi_0$,

$$\bar{w}_{\text{real}}^2(\xi) \approx 2G\bar{\Sigma}_c\xi_d \left[\int_0^{\frac{\xi}{\xi_d}} + \int_{\frac{\xi}{\xi_d}}^{\frac{\Xi_0}{\xi_d}} \right] x \exp(-x) \left[\frac{K\left(2\frac{\sqrt{x\xi/\xi_d}}{x+\xi/\xi_d}\right)}{x + \xi/\xi_d} - \frac{E\left(2\frac{\sqrt{x\xi/\xi_d}}{x-\xi/\xi_d}\right)}{x - \xi/\xi_d} \right] dx. \quad (60)$$

Note that the last term in equation (60) diverges for each integral, but that added together, the sum converges (taking its Cauchy principal value).

An approximate expression for the induced associated potential $\Psi(\xi)$ is given by

$$\Psi(\xi) \approx -\frac{\sqrt{2}c}{\pi^2} \sqrt{\frac{M_{t_0}G}{\Xi_0}} \left[\int_0^{\frac{\xi}{\xi_d}} + \int_{\frac{\xi}{\xi_d}}^{\frac{\Xi_0}{\xi_d}} \right] \frac{x \left\{ e^{-x} \text{Ei}(x) - e^x \text{Ei}(-x) \right\} K\left(2\frac{\sqrt{x\xi/\xi_d}}{x+\xi/\xi_d}\right) dx}{x + \xi/\xi_d}, \quad (61)$$

for $\xi \ll \Xi_0$. Note that the magnitude of $\Psi(\xi)$ depends critically on the upper limit of integration (if $\xi \sim \Xi_0$, a more accurate expression for $\Psi(\xi)$ may be found from equation (48)). Also note that $\Psi(\xi)$ is non-singular everywhere, and that if $\bar{\Sigma}_c$ is small enough, $|\Psi(\xi)|$ may in principle dominate over $|\Phi(\xi)|$ for the whole disk. Finally, the full rotational curve may be found approximately from the formula ($\xi \ll \Xi_0$)

$$\begin{aligned} \bar{w}_{\text{circ}}^2(\xi) \approx \bar{w}_{\text{real}}^2(\xi) + \frac{c}{\sqrt{2}\pi^2} \sqrt{\frac{M_{t_0}G}{\Xi_0}} \left[\int_0^{\frac{\xi}{\xi_d}} + \int_{\frac{\xi}{\xi_d}}^{\frac{\Xi_0}{\xi_d}} \right] x \left\{ \exp(-x) \text{Ei}(x) - \exp(x) \text{Ei}(-x) \right\} \\ \times \left[\frac{K\left(2\frac{\sqrt{x\xi/\xi_d}}{x+\xi/\xi_d}\right)}{x + \xi/\xi_d} - \frac{E\left(2\frac{\sqrt{x\xi/\xi_d}}{x+\xi/\xi_d}\right)}{x - \xi/\xi_d} \right] dx. \end{aligned} \quad (62)$$

Numerically, it is found that with increasing ξ , the integrals converge rather quickly towards a constant factor of 2π , so for ξ above about 8-10 disk lengths the induced associated contribution changes very little. This means that, for large distances, the expression for $\bar{w}_{\text{circ}}^2(\xi)$ does not depend significantly on the upper limit of integration, and that one automatically gets an asymptotically flat rotation curve. Also notice that for small distances, the integrals yield a *negative* contribution to $\bar{w}_{\text{circ}}^2(\xi)$.

4.2 The correspondence with MOND

The most basic feature of MOND is the postulation of a fundamental acceleration scale a_0 , observationally estimated to $a_0 \sim (1-2) \times 10^{-10} \text{m/s}^2$. That is, for Keplerian accelerations well below a_0 , the Newtonian acceleration $a_N = \frac{GM}{r^2}$ in a spherically symmetric system is replaced by the MOND acceleration $a_M = \frac{\sqrt{GMa_0}}{r}$. With the help of results derived in the preceding section we may now easily find a correspondence with MOND and actually *calculate* a_0 in the case of an exponential disk. From equation (62) we find that the asymptotic rotational speed and the corresponding Keplerian acceleration are given by

$$\bar{w}_\infty = \sqrt{\frac{c}{\pi} \sqrt{\frac{2M_{t_0}G}{\Xi_0}}}, \quad \Rightarrow \quad \bar{a}(\xi) = \frac{c}{\pi} \sqrt{\frac{2M_{t_0}G}{\Xi_0}} \frac{1}{\xi}, \quad \xi \gg \xi_d. \quad (63)$$

We are now able to compare equation (63) to the MOND acceleration. We then get a correspondence if

$$a_0 = \frac{2c^2}{\pi^2 \Xi_0} = \frac{2}{\pi^2} c H_0 \approx 1.5 \times 10^{-10} \text{m/s}^2, \quad (64)$$

where the expression is valid for an exponential disk at epoch t_0 (with H_0 as the corresponding Hubble parameter), and where the final, numerical result is valid for the present epoch. (Note that a_0 is predicted to decrease with time, since the Hubble parameter decreases as the inverse of cosmic epoch.) This means that there is a correspondence between the model of a thin disk presented in this paper and MOND for the asymptotically flat part of the rotation curve. In particular, the fact that MOND successfully fits the observed baryonic Tully-Fisher relation [11] means that quasi-metric gravity does as well. That is, from equation (63) we get that

$$\bar{w}_\infty^4 = \frac{2M_{t_0}Gc^2}{\pi^2\Xi_0} = \frac{2M_{t_0}G}{\pi^2}cH_0, \quad \Rightarrow \quad M_{t_0} = \frac{\pi^2\Xi_0}{2Gc^2}\bar{w}_\infty^4 = \frac{\pi^2}{2GcH_0}\bar{w}_\infty^4, \quad (65)$$

and since the constant of proportionality in the relation $M_{t_0} \propto \bar{w}_\infty^4$ is measured to be about $50 \text{ M}_\odot \text{s}^4/\text{km}^4$ [11], it is straightforward to check that this result is in very good agreement with equation (65) (for the present epoch). This justifies the definition of $\bar{\sigma}(-1)$ made in equation (32). Note that, what enters into equation (65) is the active gravitational mass M_t (at epoch t_0); this increases linearly with cosmic epoch. Moreover, since the Hubble parameter decreases as the inverse of cosmic epoch, this means that \bar{w}_∞ is constant; i.e., that it does not evolve with epoch. In other words, the quasi-metric model predicts that the sizes of metrically stationary galactic disks increase with epoch, but such that rotational speeds are unaffected.

Moreover, a FO at an arbitrary epoch t_1 , using his locally measured values $\frac{t_1}{t_0}G$ and $\frac{t_0}{t_1}H_0$ of the gravitational “constant” and the Hubble parameter, respectively, is predicted to measure the same slope of the local baryonic Tully-Fisher relation as will a FO at epoch t_0 . This means that the slope of the baryonic Tully-Fisher relation is predicted not to depend on epoch, and this seems to be confirmed by observations [12, 13]. (The analyses presented in these papers are based on the standard cosmological framework, so some inconsistency with the QMF might be expected. But as long as theory-dependent gravitational physics (i.e., inconsistent with the QMF) is not used to infer masses, this should not matter too much.)

5 The general weak-field, axisymmetric case

Having analysed a flat disk, the question now is if the results of the previous sections can be extended to the general, weak-field axisymmetric case. To answer that question, we will follow the standard method of breaking up an arbitrary axisymmetric matter distribution into a series of concentric, spherical shells (of negligible thickness) with corresponding surface densities. The total potential at a given point will then be the sum of

the potentials due to the collection of shells. Except for the limitation to axial symmetry, this procedure is the counterpart to the derivation of a general multipole expansion for the Newtonian case as given in [7], which we will follow closely.

The mathematical task is then to solve equation (7) (with $V = 0$) interior respectively exterior to a given isolated shell, subject to suitable boundary conditions. Since equation (7) becomes separable we may write solutions in the form $\bar{B}(\rho, \theta) = 1 + 2c^{-2}\Phi(\rho, \theta)$, where $c^{-2}\Phi(\rho, \theta) \equiv -\bar{F}(\rho)\bar{G}(\theta)$. General mode solutions $\bar{F}_{\beta\pm}(\rho)$, $\bar{G}_{\beta\pm}(\theta)$ are given by equations (A.3), (A.4) in appendix A. We will select specific mode solutions $\bar{F}_n(\rho)$, $\bar{G}_n(\theta)$, where $n = 0, 1, 2, \dots$ are whole non-negative numbers, such that

$$c^{-2}\Phi_n(\rho, \theta) = -\sqrt{\frac{\Xi_0}{\rho}} \left[A_n P_{\frac{1}{2}}^{-n-\frac{1}{2}} \left(\sqrt{1 - \frac{\rho^2}{\Xi_0^2}} \right) + B_n P_{\frac{1}{2}}^{n+\frac{1}{2}} \left(\sqrt{1 - \frac{\rho^2}{\Xi_0^2}} \right) \right] P_n(\cos\theta), \quad (66)$$

where A_n and B_n are constants. We now construct solutions $\Phi_{\text{int}}(\rho, \theta)$, $\Phi_{\text{ext}}(\rho, \theta)$ respectively interior and exterior to a shell located at $\rho = \rho_s$, by summing up suitable mode solutions. That is, by requiring that the interior solution should be regular at the center of the shell we find that

$$c^{-2}\Phi_{\text{int}}(\rho, \theta) = -\sqrt{\frac{\Xi_0}{\rho}} \sum_{n=0}^{\infty} A_n P_{\frac{1}{2}}^{-n-\frac{1}{2}} \left(\sqrt{1 - \frac{\rho^2}{\Xi_0^2}} \right) P_n(\cos\theta), \quad \rho < \rho_s. \quad (67)$$

The exterior solution is found by requiring that $\Phi_{\text{ext}}(\Xi_0, \theta)$ must vanish. So, for $\rho > \rho_s$,

$$c^{-2}\Phi_{\text{ext}}(\rho, \theta) = -\sqrt{\frac{\Xi_0}{\rho}} \sum_{n=0}^{\infty} \left[D_n P_{\frac{1}{2}}^{-n-\frac{1}{2}} \left(\sqrt{1 - \frac{\rho^2}{\Xi_0^2}} \right) + B_n P_{\frac{1}{2}}^{n+\frac{1}{2}} \left(\sqrt{1 - \frac{\rho^2}{\Xi_0^2}} \right) \right] P_n(\cos\theta), \quad (68)$$

where the constants D_n are given by the expression $D_n = -B_n P_{\frac{1}{2}}^{n+\frac{1}{2}}(0) / P_{\frac{1}{2}}^{-n-\frac{1}{2}}(0)$, which vanishes for all even n (see equation (A.5)).

Now the surface density $\bar{\Sigma}_s(\rho_s, \theta)$ of the shell can be expressed as a sum of modes; this yields counterparts to equations (20)-(22) valid for a thin disk. We thus have

$$\bar{\Sigma}_s(\rho_s, \theta) = \frac{1}{2} \sum_{n=0}^{\infty} (2n+1) \bar{\Sigma}_{sn}(\rho_s) P_n(\cos\theta), \quad \bar{\Sigma}_{sn}(\rho_s) = \int_0^\pi \bar{\Sigma}_s(\rho_s, \theta') P_n(\cos\theta') \sin\theta' d\theta'. \quad (69)$$

To determine the constants A_n and B_n , we now apply Gauss' theorem across the shell. Assuming a weak gravitational field ($\bar{B} \approx 1$) we then have

$$\sqrt{1 - \frac{\rho_s^2}{\Xi_0^2}} \left[\left(\frac{\partial \Phi_{\text{ext}}}{\partial \rho} \right)_{\rho=\rho_s} - \left(\frac{\partial \Phi_{\text{int}}}{\partial \rho} \right)_{\rho=\rho_s} \right] = 4\pi G \bar{\Sigma}_s(\rho_s, \theta). \quad (70)$$

Since the potential must be continuous over the shell, we have that $\Phi_{\text{ext}}(\rho_s, \theta) = \Phi_{\text{int}}(\rho_s, \theta)$. Furthermore, due to the orthogonality of the Legendre polynomials, this means that

$$\left(A_n + \frac{P_{\frac{1}{2}}^{n+\frac{1}{2}}(0)}{P_{\frac{1}{2}}^{-n-\frac{1}{2}}(0)}B_n\right)P_{\frac{1}{2}}^{-n-\frac{1}{2}}\left(\sqrt{1-\frac{\rho_s^2}{\Xi_0^2}}\right) = B_nP_{\frac{1}{2}}^{n+\frac{1}{2}}\left(\sqrt{1-\frac{\rho_s^2}{\Xi_0^2}}\right). \quad (71)$$

A second equation relating the constants A_n and B_n can be found from equation (70) by inserting equation (69). Calculating the derivatives and using recurrence formulae valid for Legendre functions, one finds that (since the Legendre polynomials are orthogonal)

$$\begin{aligned} (1-n^2)P_{\frac{1}{2}}^{n-\frac{1}{2}}\left(\sqrt{1-\frac{\rho_s^2}{\Xi_0^2}}\right)B_n + \left(A_n + \frac{P_{\frac{1}{2}}^{n+\frac{1}{2}}(0)}{P_{\frac{1}{2}}^{-n-\frac{1}{2}}(0)}B_n\right)P_{\frac{1}{2}}^{-n+\frac{1}{2}}\left(\sqrt{1-\frac{\rho_s^2}{\Xi_0^2}}\right) \\ = (2n+1)\sqrt{\frac{\rho_s}{\Xi_0}}\frac{\bar{\Sigma}_{sn}(\rho_s)}{\bar{\Sigma}_*}. \end{aligned} \quad (72)$$

One may find explicit expressions for A_n and B_n from equations (71) and (72). We get

$$A_n = (2n+1)\sqrt{\frac{\rho_s}{\Xi_0}}\frac{\bar{\Sigma}_{sn}(\rho_s)}{\bar{\Sigma}_*}g_n(\rho_s), \quad B_n = (2n+1)\sqrt{\frac{\rho_s}{\Xi_0}}\frac{\bar{\Sigma}_{sn}(\rho_s)}{\bar{\Sigma}_*}f_n(\rho_s), \quad (73)$$

$$f_n(\rho_s) \equiv \frac{P_{\frac{1}{2}}^{-n-\frac{1}{2}}(y_s)}{(1-n^2)P_{\frac{1}{2}}^{-n-\frac{1}{2}}(y_s)P_{\frac{1}{2}}^{n-\frac{1}{2}}(y_s) + P_{\frac{1}{2}}^{n+\frac{1}{2}}(y_s)P_{\frac{1}{2}}^{-n+\frac{1}{2}}(y_s)}, \quad y_s \equiv \sqrt{1-\frac{\rho_s^2}{\Xi_0^2}}, \quad (74)$$

$$g_n(\rho_s) \equiv \left[\frac{P_{\frac{1}{2}}^{n+\frac{1}{2}}(y_s)}{P_{\frac{1}{2}}^{-n-\frac{1}{2}}(y_s)} - \frac{P_{\frac{1}{2}}^{n+\frac{1}{2}}(0)}{P_{\frac{1}{2}}^{-n-\frac{1}{2}}(0)} \right] f_n(\rho_s). \quad (75)$$

Equations (73)-(75) may now be inserted into equations (67), (68) to get complete expressions for $\Phi_{\text{int}}(\rho, \theta)$ and $\Phi_{\text{ext}}(\rho, \theta)$. This yields the potential generated by a single, infinitesimally thin shell.

To evaluate the potential generated by an entire collection of shells filling space, we let $\delta\bar{\Sigma}_s(\rho_s, \theta)$ and $\delta\bar{\Sigma}_{sn}(\rho_s)$ denote the relevant quantities for a shell lying between ρ_s and $\rho_s + \delta\rho_s$. From equation (69) we then have (inserting $\delta\bar{\Sigma}_s(\rho_s, \theta') = \bar{\varrho}_m(\rho_s, \theta')\delta\rho_s/\sqrt{1-\frac{\rho_s^2}{\Xi_0^2}}$)

$$\delta\bar{\Sigma}_{sn}(\rho_s) = \int_0^\pi \bar{\varrho}_m(\rho_s, \theta')P_n(\cos\theta')\sin\theta'd\theta' \frac{\delta\rho_s}{\sqrt{1-\frac{\rho_s^2}{\Xi_0^2}}} \equiv \frac{2\bar{\varrho}_{mn}(\rho_s)\delta\rho_s}{\sqrt{1-\frac{\rho_s^2}{\Xi_0^2}}}, \quad (76)$$

where $\bar{\varrho}_m(\rho_s, \theta)$ is the coordinate volume density of (active) mass [4, 5]. Substituting equation (76) into the complete expressions for the corresponding potentials $\delta\Phi_{\text{int}}(\rho, \theta)$ and $\delta\Phi_{\text{ext}}(\rho, \theta)$ and integrating over ρ_s , we find that (with $x_s \equiv \frac{\rho_s}{\Xi_0}$, $y \equiv \sqrt{1 - \frac{\rho^2}{\Xi_0^2}}$ and $\bar{\varrho}_{m*} \equiv \frac{c^2}{4\pi G \Xi_0^2}$)

$$\begin{aligned} c^{-2}\Phi(\rho, \theta) &= c^{-2} \sum_{\rho_s=0}^{\rho} \delta\Phi_{\text{ext}}(\rho, \theta) + c^{-2} \sum_{\rho_s=\rho}^{\Xi_0} \delta\Phi_{\text{int}}(\rho, \theta) \\ &= -\sqrt{\frac{\Xi_0}{\rho}} \sum_{n=0}^{\infty} \frac{(2n+1)}{\bar{\varrho}_{m*}} P_n(\cos\theta) \left[\left\{ P_{\frac{1}{2}}^{n+\frac{1}{2}}(y) - \frac{P_{\frac{1}{2}}^{n+\frac{1}{2}}(0)}{P_{\frac{1}{2}}^{-n-\frac{1}{2}}(0)} P_{\frac{1}{2}}^{-n-\frac{1}{2}}(y) \right\} \right. \\ &\quad \left. \times \int_0^{\frac{\rho}{\Xi_0}} \frac{\sqrt{x_s} f_n(x_s) \bar{\varrho}_{mn}(x_s) dx_s}{\sqrt{1-x_s^2}} + P_{\frac{1}{2}}^{-n-\frac{1}{2}}(y) \int_{\frac{\rho}{\Xi_0}}^1 \frac{\sqrt{x_s} g_n(x_s) \bar{\varrho}_{mn}(x_s) dx_s}{\sqrt{1-x_s^2}} \right]. \end{aligned} \quad (77)$$

Equation (77) is the axisymmetric counterpart to the similar general multipole expansion formula in Newtonian theory; this is given explicitly in [7]. These two expressions have a correspondence in the limit $\Xi_0 \rightarrow \infty$. For example, the potential at $\rho = 0$ is given by $c^{-2}\Phi(0) = -\frac{1}{\bar{\varrho}_{m*}} \int_0^1 x_s \bar{\varrho}_{m0}(x_s) dx_s$, the same as the Newtonian expression in said limit. This means that as expected, equation (77) can not describe DM-effects.

Moreover, due to the boundary condition at $\rho = \Xi_0$, there is no obvious way to define an associated (volume) density as a counterpart to equation (32). But if no associated density can be defined, one cannot motivate a definition similar to equation (43) for the induced associated (volume) density. So it would seem that, the explanation of DM-effects presented in this paper should exclusively be connected to disks or other structures that do not satisfy the boundary condition $\Phi(\Xi_0) = 0$, answering the question asked at the beginning of this section.

However, DM-effects are seen in galaxies other than spiral galaxies. In particular, observations indicate that dwarf spheroidal galaxies are the most DM-dominated systems ever found [14]. Yet the general observational status for the existence of DM in elliptical galaxies is more complicated than for spiral galaxies, since some ordinary elliptical galaxies apparently lack significant amounts of it [15], while others seem to have plenty [16]. It still remains the challenging task of explaining these observations without DM.

6 Gravitational lensing

For a sufficiently weak gravitational field, rotation curves can be calculated accurately enough using the auxiliary metric family $\bar{\mathbf{g}}_t$ rather than the full “physical” metric family \mathbf{g}_t . However, when calculating deflection of light, or gravitational lensing, it is not suf-

ficient to know $\bar{\mathbf{g}}_t$, even if the gravitational field is weak. But to calculate gravitational lensing, it is fortunately not necessary to know \mathbf{g}_t in full; as we shall see, a suitable approximation will be sufficient.

The general formulae describing the transformation $\bar{\mathbf{g}}_t \rightarrow \mathbf{g}_t$ are given by [2,3]

$$g_{(t)00} = \left(1 - \frac{v^2}{c^2}\right)^2 \bar{g}_{(t)00}, \quad (78)$$

$$g_{(t)0j} = \left(1 - \frac{v^2}{c^2}\right) \left[\bar{g}_{(t)0j} + \frac{t}{t_0} \frac{2v}{1 - \frac{v}{c}} (\bar{e}_b^i \bar{N}_i) \bar{e}_j^b \right], \quad (79)$$

$$g_{(t)ij} = \bar{g}_{(t)ij} + \frac{t^2}{t_0^2} \frac{4v}{(1 - \frac{v}{c})^2} \bar{e}_i^b \bar{e}_j^b, \quad (80)$$

where $\bar{\mathbf{e}}_b \equiv \frac{t_0}{t} \bar{e}_b^i \frac{\partial}{\partial x^i}$ and $\bar{\mathbf{e}}^b \equiv \frac{t}{t_0} \bar{e}_i^b dx^i$ are unit vector and covector fields, respectively, along the 3-vector field $\mathbf{b}_{\mathcal{F}}$ found from the set of linear algebraic equations

$$\left[\bar{a}_{\mathcal{F}|k}^k + c^{-2} \bar{a}_{\mathcal{F}k} \bar{a}_{\mathcal{F}}^k \right] b_{\mathcal{F}}^j - \left[\bar{a}_{\mathcal{F}|k}^j + c^{-2} \bar{a}_{\mathcal{F}k} \bar{a}_{\mathcal{F}}^j \right] b_{\mathcal{F}}^k - 2 \bar{a}_{\mathcal{F}}^j = 0, \quad (81)$$

and where $v \equiv c^{-1} \sqrt{\bar{a}_{\mathcal{F}k} \bar{a}_{\mathcal{F}}^k b_{\mathcal{F}i} b_{\mathcal{F}}^i}$. For a weak gravitational field and for distances much smaller than Ξ_0 (at epoch t_0), we have that $v \ll c$, so to a good approximation we may neglect terms of order 2 or higher in the small quantity v/c . This means that we may set (assuming $\bar{N}_i \approx 0$)

$$g_{(t)00} \approx \bar{g}_{(t)00}, \quad g_{(t)0j} \approx 0, \quad g_{(t)ij} \approx \bar{g}_{(t)ij} + 4 \frac{t^2}{t_0^2} \frac{v}{c} \bar{e}_i^b \bar{e}_j^b. \quad (82)$$

To get explicit formulae, it is easiest to solve the set of equations (81) using standard methods, in spherical coordinates. One may then transform to cylindrical coordinates. If one additionally assumes a weak field in vacuum, and neglects all terms proportional to Ξ_0^{-2} , one may give $b_{\mathcal{F}}^\xi$ and $b_{\mathcal{F}}^z$ as series expansions in z . Terminating the series after the linear term, yields (using the fact that in vacuum, $\bar{B}_{,zz} \approx -\bar{B}_{,\xi\xi} - \frac{1}{\xi} \bar{B}_{,\xi}$ from equation (10))

$$b_{\mathcal{F}}^\xi \approx \frac{-2 \left[(\bar{B}_{,\xi\xi} + \frac{1}{\xi} \bar{B}_{,\xi}) \bar{B}_{,\xi} + \bar{B}_{,\xi z} \bar{B}_{,z} \right]}{\left(\bar{B}_{,\xi\xi} + \frac{1}{\xi} \bar{B}_{,\xi} \right) \bar{B}_{,\xi\xi} + (\bar{B}_{,\xi z})^2} + \frac{z}{\xi} b_{\mathcal{F}}^z + O(z^2), \quad (83)$$

$$b_{\mathcal{F}}^z \approx \frac{2 \left[\bar{B}_{,\xi\xi} \bar{B}_{,z} - \bar{B}_{,\xi z} \bar{B}_{,\xi} \right]}{\left(\bar{B}_{,\xi\xi} + \frac{1}{\xi} \bar{B}_{,\xi} \right) \bar{B}_{,\xi\xi} + (\bar{B}_{,\xi z})^2} + O(z^2), \quad (84)$$

where the derivatives may be calculated approximately to first order in $|z|$ from the

expression

$$\begin{aligned}\bar{B}(\xi, z) &\approx 1 + \frac{2}{c^2} \left(\Phi(\xi) + \Psi(\xi) \right) + \frac{4\pi G}{c^2} \left(\bar{\Sigma}(\xi) + \bar{\sigma}^i(\xi) \right) |z| \\ &- \frac{1}{c^2} \left[\Phi_{,\xi\xi} + \frac{1}{\xi} \Phi_{,\xi} + \Psi_{,\xi\xi} + \frac{1}{\xi} \Psi_{,\xi} \right] z^2 + O(|z|^3),\end{aligned}\quad (85)$$

obtained from equation (51) and the approximation for $\bar{B}_{,zz}$ shown above. Furthermore, since the quantities \bar{e}_ξ^b and \bar{e}_z^b may be found to the relevant accuracy from the definition

$$\bar{e}_j^b = \bar{h}_{(t_0)ij} \bar{e}_b^i \equiv \bar{h}_{(t_0)ij} \frac{b_{\mathcal{F}}^i}{|\mathbf{b}_{\mathcal{F}}|} \equiv \frac{b_{\mathcal{F}j}}{|\mathbf{b}_{\mathcal{F}}|}, \quad |\mathbf{b}_{\mathcal{F}}| \equiv \sqrt{b_{\mathcal{F}k} b_{\mathcal{F}}^k}, \quad (86)$$

we may also calculate the relevant approximation of the desired quantities $g_{(t)ij}$ from equation (82), i.e.,

$$g_{(t)ij} \approx \bar{g}_{(t)ij} + 4 \frac{t^2}{t_0^2} \frac{v}{c} \bar{e}_i^b \bar{e}_j^b \approx \bar{g}_{(t)ij} + 4 \frac{t^2}{t_0^2} \sqrt{\frac{\bar{B}_{,k} \bar{B}_{,k}}{b_{\mathcal{F}s} b_{\mathcal{F}}^s}} b_{\mathcal{F}i} b_{\mathcal{F}j}. \quad (87)$$

From the geodesic equation, we may now calculate the gravitational bending of a light ray grazing the plane of the disk, since in this case, $|z|$ is small enough so that the approximation given in equation (85) is valid. To deal with the opposite situation, where the light path is nearly orthogonal to the disk plane, the approximations given in equations (83)-(85) may not be sufficient; then one must include terms of higher order in $|z|$.

Finally, we note that in GR, the weak field form of the metric outside the disk is assumed to take the form

$$ds^2 = -\left(1 + \frac{2}{c^2} \Phi_N\right) (dx^0)^2 + \left(1 - \frac{2}{c^2} \Phi_N\right) (dx^2 + dy^2 + dz^2), \quad (88)$$

where Φ_N is the Newtonian potential for the sum of visible and DM. But in general, the quantities $g_{(t_0)ij}$ as found from equation (87) will not correspond to their counterparts given by equation (88). This means that any observationally based mapping of DM distributions using gravitational lensing, assuming the weak field approximation obtained from GR, is explicitly theory-dependent and may give misleading results.

7 Discussion

For many years, it has been known that galactic dynamics is incompatible with a straightforward application of Newtonian theory to visible matter. However, the most glaring discrepancies between observations and theory can be removed by assuming the existence of galactic DM. Since the introduction of DM can be done without making radical

changes to the standard theoretical framework underlying mainstream astrophysics, this is currently the preferred approach. On the other hand, MOND interpreted as an empirical recipe, has an impressive successful record when predicting galactic phenomena. But the connection between MOND and fundamental physics has been unclear so far.

Contrary to other approaches, the explanation of some galactic phenomena given in this paper has not assumed any empirical aspect of galactic dynamics as input to the model. Rather, while formally belonging to the “modified gravity” category, the model comes directly from the weak field approximation of the QMF, without any extra modifications of the theory. (The only extra assumption made, is that the induced associated surface density $\bar{\sigma}^i(u)$ should be treated as a gravitating source in the field equations.) The main reason why this is possible is that according to the QMF, the Universe is finite and “small”, so that boundary conditions depend crucially on the shape of the matter distribution. (A finite and “small” Universe is incompatible with cosmological data as interpreted within the standard framework, therefore the DM explanation given here is compatible with the weak field limit of the QMF but not with the weak field limit of standard cosmology.) In particular, for a flat disk we found that $\Phi(u = -1) \neq 0$ in equation (29), defining a specific velocity scale dependent on the total mass of the disk. To be able to define the associated surface density $\bar{\sigma}(u)$ in equation (32), it is essential that this velocity scale does not vanish. However, by construction, it does vanish for the general axisymmetric matter distribution considered in section 5, so flat disks seem to be an exceptional case. Anyway, that case should apply to all thin disks (even if they are not exactly flat).

The other crucial feature is the existence of an induced matter surface density $\bar{\Sigma}^i(u)$ directly dependent on the real matter surface density as shown in equation (42). This, together with the existence of $\bar{\sigma}(u)$, is sufficient to define the induced associated surface density $\bar{\sigma}^i(u)$ playing the role of DM. The introduction of $\bar{\sigma}^i(u)$ may seem “contrived” to some people. However, the facts that the induced associated surface density corresponding to an exponential disk automatically yields an asymptotically flat rotation curve and a correspondence with MOND are *calculated* results, and not put in by hand *a priori*. It was not at all obvious that these results would be possible.

It has been claimed [17] that observations of gravitational lensing in the colliding clusters 1E0657-56 (the Bullet cluster) represent a “direct” detection of DM, since these observations indicate that the DM is associated with the regions containing the field galaxies rather than with the regions containing the more massive gas making up the bulk of the cluster. However, as we have seen, in the QMF, the existence of any sort of “phantom” matter density similar to $\bar{\sigma}^i(u)$, playing the role of DM, is crucially dependent

on the *shape* of the matter distribution. This means that, e.g., a large nearly spherical or spheroidal mass distribution of gas, should not necessarily be associated with much DM as inferred from gravitational lensing. So, since the colliding gas clouds in the Bullet cluster shown in [17] do not seem to have shapes that could in any way resemble disks, this might be a natural explanation of why they do not seem to be associated with much DM. But of course, further justification of this explanation will be necessary, together with an explanation of why dwarf spheroidals and some elliptic galaxies seem to be DM-dominated. Anyway, the mere existence of such an explanation shows that the interpretation of the Bullet cluster observations is not theory-independent, so citing them as definite evidence of the existence of DM is unjustified.

In light of the results found in this paper, it seems that some lines of argument favouring DM over modified gravity have been shown to be invalid. First, there now exists a natural correspondence between MOND and fundamental physics. This indicates that at least some galactic phenomenology has its basis in *geometry* rather than in the properties of some unknown exotic particle. Second, while the correspondence with MOND works for spiral galaxies, this does not imply that such a correspondence is necessarily valid for other types of galaxies or galaxy clusters. This means that it may be possible to share MOND’s successes but not necessarily its failures. (Further work should be done to see if this is indeed the case.) Third, some specific observations (of, e.g., the Bullet cluster) have been hailed as the ultimate “direct proof” that DM prevails over modified gravity. But since standard interpretations of such observations are model-dependent, this is not true.

There is now near consensus that galactic phenomenology has its basis in some suitable nonbaryonic particle, despite the fact that no such particle has ever been detected directly. Furthermore, many astrophysicists are of the opinion that this is not necessary, i.e., that indirect observations are sufficient to rule out any alternative explanations. This means that for many astrophysicists, taking the focus off DM and recognising the merits of modified gravity, is a radical move that is out of the question, regardless of how contrived and *ad hoc* explanations based on DM might be. This is a perilous attitude, since interpretations of indirect observations are often crucially theory-dependent. Another example of this is interpretations of some indirect observations made in the solar system that are also presented as indisputable facts, even if alternative interpretations are not ruled out [4]. So it would seem that, for at least some parts of astrophysics, there is a serious lack of critical assessment of basic assumptions underlying mainstream knowledge. One may hope that this will change in the future.

References

- [1] A. Aguirre, C.P. Burgess, A. Friedland, and D. Nolte, *Class. Quantum Grav.* **18**, R223 (2001).
- [2] D. Østvang, *Grav. & Cosmol.* **11**, 205 (2005) (gr-qc/0112025).
- [3] D. Østvang, *Doctoral thesis*, NTNU, 2001 (gr-qc/0111110).
- [4] D. Østvang, *Grav. & Cosmol.* **13**, 1 (2007) (gr-qc/0201097).
- [5] D. Østvang, *Acta Physica Polonica B* **39**, 1849 (2008) (gr-qc/0510085).
- [6] A. Toomre, *Astrophys. Journ.* **138**, 385 (1963).
- [7] J. Binney and S. Tremaine, *Galactic dynamics*, Princeton University Press (1987).
- [8] M. Abramowitz and I.A. Stegun, *Handbook of mathematical functions*, Dover (1972).
- [9] A.J. Jerri, *Integral and discrete transformations with applications and error analysis*, Marcel Dekker Inc. (1992).
- [10] G.N. Watson, *J. London Math. Soc.* **8**, 289 (1933).
- [11] S.S. McGaugh, *Astrophys. Journ.* **632**, 859 (2005).
- [12] H. Flores *et al.*, *A&A* **455**, 107 (2006) (astro-ph/0603563).
- [13] M. Puech *et al.*, *A&A* **510**, A68 (2010) (arXiv:0903.3961).
- [14] S.S. McGaugh and J. Wolf, *Astrophys. Journ.* **722**, 248 (2010) (arXiv:1003.3448).
- [15] A. Romanowsky *et al.*, *Science* **301**, 1696 (2003).
- [16] T. Bridges *et al.*, *MNRAS* **373**, 157 (2006).
- [17] D. Clowe *et al.*, *Astrophys. Journ.* **648**, L109 (2006) (astro-ph/0608407).

A Boundary behaviour of mode solutions

In this Appendix, we list mode solutions of the vacuum field equations for axially symmetric, metrically static, isolated sources with various “pure” values of their multipole moments. As we shall see, these solutions can be classified into two groups; those that admit the boundary condition $\bar{B}(\Xi_0) = 1$ and those that do not.

Starting with equation (7), we may set $\bar{V} = 0$ for a non-rotating source (this is a good approximation for slowly rotating sources and weak gravitational fields also). Equation (7) then becomes separable, i.e., solutions of it can be written in the form $\bar{B}(\rho, \theta) = 1 - 2\bar{F}(\rho)\bar{G}(\theta)$. The new functions \bar{F} and \bar{G} must then satisfy the ordinary differential equations

$$\left(1 - \frac{\rho^2}{\Xi_0^2}\right) \frac{d^2 \bar{F}}{d\rho^2} + \frac{2}{\rho} \left(1 - \frac{3\rho^2}{2\Xi_0^2}\right) \frac{d\bar{F}}{d\rho} - \frac{\beta}{\rho^2} \bar{F} = 0, \quad (\text{A.1})$$

$$\frac{d^2 \bar{G}}{d\theta^2} + \cot\theta \frac{d\bar{G}}{d\theta} + \beta \bar{G} = 0, \quad (\text{A.2})$$

where β is some complex-valued constant. Restricting β to be real and requiring that $\beta \geq -\frac{1}{4}$, the general solutions of equations (A.1) and (A.2) may be written in the form

$$\bar{F}_{\beta\pm}(\rho) = \sqrt{\frac{\Xi_0}{\rho}} \left[c_{\beta\pm} P_{\frac{1}{2}}^{\pm\sqrt{\beta+\frac{1}{4}}} \left(\sqrt{1 - \frac{\rho^2}{\Xi_0^2}} \right) + c_{\beta\pm}^i Q_{\frac{1}{2}}^{\pm\sqrt{\beta+\frac{1}{4}}} \left(\sqrt{1 - \frac{\rho^2}{\Xi_0^2}} \right) \right], \quad (\text{A.3})$$

$$\bar{G}_{\beta\pm}(\theta) = C_{\beta\pm} P_{\pm\sqrt{\beta+\frac{1}{4}}-\frac{1}{2}}(\cos\theta) + C_{\beta\pm}^i Q_{\pm\sqrt{\beta+\frac{1}{4}}-\frac{1}{2}}(\cos\theta), \quad (\text{A.4})$$

where $P_\nu^\mu(x)$, $Q_\nu^\mu(x)$ are the usual associated Legendre functions of the first and second kind, respectively, and where $c_{\beta\pm}$, $c_{\beta\pm}^i$, $C_{\beta\pm}$ and $C_{\beta\pm}^i$ are (dimensionless) constants. Note that the solutions (A.3) and (A.4) are real-valued functions, and so are $P_\nu^\mu(x)$, $Q_\nu^\mu(x)$ (for real μ, ν) since they are defined on the cut $(-1, 1)$ by averaging the relevant limiting values of the corresponding complex-valued quantities $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$ [8].

We will now find solutions (A.3) admitting the boundary condition $\bar{F}_{\beta\pm}(\Xi_0) = 0$.

To do that, we first notice that, since $P_{\frac{1}{2}}^{\frac{1}{2}}(\sqrt{1 - \frac{\rho^2}{\Xi_0^2}}) = \sqrt{\frac{2\Xi_0(1 - \frac{\rho^2}{\Xi_0^2})}{\pi\rho}}$ and since only the trivial constant solution is obtained from $P_{\frac{1}{2}}^{-\frac{1}{2}}(\sqrt{1 - \frac{\rho^2}{\Xi_0^2}}) = \sqrt{\frac{2\rho}{\pi\Xi_0}}$, choosing the $+$ -sign, with $C_{0+} = 1$ and a suitable choice of c_{0+} , the value $\beta = 0$ corresponds to the unique spherically symmetric solution found in ref. [4]. That solution is unique since the function $Q_{\frac{1}{2}}^{-\frac{1}{2}}(x)$ differs from $P_{\frac{1}{2}}^{\frac{1}{2}}(x)$ only by a numerical factor, and since $Q_{\frac{1}{2}}^{\frac{1}{2}}(\sqrt{1 - \frac{\rho^2}{\Xi_0^2}})$ again only yields the trivial constant solution. Besides, since the function $Q_0(\cos\theta)$ has singularities whenever $\theta = 0$ or $\theta = \pi$, the corresponding solution cannot be physical (but might be considered as a mode solution). Moreover, $P_{-1}(\cos\theta) = P_0(\cos\theta)$ gives nothing new, and since $Q_{-1}(\cos\theta)$ is undefined, that expression cannot be considered at all.

Secondly we notice that, since choosing $\beta = 2$ yields $P_{\frac{1}{2}}^{\frac{3}{2}}(\sqrt{1 - \frac{\rho^2}{\Xi_0^2}}) = -\sqrt{\frac{2\Xi_0^3}{\pi\rho^3}}$ and $P_{\frac{1}{2}}^{-\frac{3}{2}}(\sqrt{1 - \frac{\rho^2}{\Xi_0^2}}) = \sqrt{\frac{\Xi_0^3}{2\pi\rho^3}} \left(\arccos(\sqrt{1 - \frac{\rho^2}{\Xi_0^2}}) - \frac{\rho}{\Xi_0} \sqrt{1 - \frac{\rho^2}{\Xi_0^2}} \right)$ which do not fulfil the required boundary condition, and since $Q_{\frac{1}{2}}^{\frac{3}{2}}(x) \equiv 0$ on the cut ($Q_{\frac{1}{2}}^{-\frac{3}{2}}(x)$ is undefined), the next suitable solution is found by choosing $\beta = 6$. This is so, since even though $P_{\frac{1}{2}}^{-\frac{5}{2}}(\sqrt{1 - \frac{\rho^2}{\Xi_0^2}})$ does not fulfil the required boundary condition, $P_{\frac{1}{2}}^{\frac{5}{2}}(\sqrt{1 - \frac{\rho^2}{\Xi_0^2}}) = 3\sqrt{\frac{2\Xi_0^5(1 - \frac{\rho^2}{\Xi_0^2})}{\pi\rho^5}}$ obviously does. This solution was found in ref. [5] and corresponds to a pure quadrupole field since it involves the Legendre polynomial $P_{-3}(\cos\theta) = P_2(\cos\theta)$ (also, $Q_2(\cos\theta)$ is singular and the corresponding mode solution thus unphysical, and $Q_{-3}(\cos\theta)$ is undefined). Moreover,

given the required boundary condition, this solution is also unique since $Q_{\frac{1}{2}}^{\frac{5}{2}}(x) \equiv 0$ on the cut ($Q_{\frac{1}{2}}^{-\frac{5}{2}}(x)$ is undefined).

Similarly, since the subsequent suitable solution is found by choosing $\beta = 20$ and the $+$ -sign, corresponding to a pure octopole field involving the Legendre polynomial $P_{-5}(\cos\theta) = P_4(\cos\theta)$, it would seem that all suitable solutions are given by choosing the $+$ -sign and $\beta = 2n(2n+1)$, $n = 0, 1, 2, \dots$, corresponding to pure even multipole fields involving the Legendre polynomials $P_{-2n-1}(\cos\theta) = P_{2n}(\cos\theta)$. With the required boundary condition, these solutions are also unique, since the functions $P_{\frac{1}{2}}^{-m-\frac{1}{2}}(\sqrt{1-\frac{\rho^2}{\Xi_0^2}})$, $m = 0, 1, 2, \dots$, do not admit it, and since $Q_{\frac{1}{2}}^{m+\frac{3}{2}}(x) \equiv 0$ on the cut (found by using the recurrence relation $Q_{\nu}^{\mu+2}(x) + 2(\mu+1)x(1-x^2)^{-1/2}Q_{\nu}^{\mu+1}(x) + (\nu-\mu)(\nu+\mu+1)Q_{\nu}^{\mu}(x) = 0$). Also, $Q_{\frac{1}{2}}^{-m-\frac{3}{2}}(x)$ and $Q_{-m-1}(\cos\theta)$ are all undefined. Finally, since $Q_{2n}(\cos\theta)$ is singular for all n , the corresponding mode solutions are unphysical.

That the above extrapolation is correct, can be checked by using the formulae [8]

$$P_{\frac{1}{2}}^{\pm\sqrt{\beta+\frac{1}{4}}}(0) = 2^{\pm\sqrt{\beta+\frac{1}{4}}}\pi^{-\frac{1}{2}}\cos\left[\frac{\pi}{2}\left(\frac{1}{2}\pm\sqrt{\beta+\frac{1}{4}}\right)\right]\frac{\Gamma\left(\frac{3}{4}\pm\frac{1}{2}\sqrt{\beta+\frac{1}{4}}\right)}{\Gamma\left(\frac{5}{4}\mp\frac{1}{2}\sqrt{\beta+\frac{1}{4}}\right)}, \quad (\text{A.5})$$

$$Q_{\frac{1}{2}}^{\pm\sqrt{\beta+\frac{1}{4}}}(0) = -2^{\pm\sqrt{\beta+\frac{1}{4}}-1}\pi^{\frac{1}{2}}\sin\left[\frac{\pi}{2}\left(\frac{1}{2}\pm\sqrt{\beta+\frac{1}{4}}\right)\right]\frac{\Gamma\left(\frac{3}{4}\pm\frac{1}{2}\sqrt{\beta+\frac{1}{4}}\right)}{\Gamma\left(\frac{5}{4}\mp\frac{1}{2}\sqrt{\beta+\frac{1}{4}}\right)}. \quad (\text{A.6})$$

Since the reciprocal gamma function $1/\Gamma(x)$ possesses simple zeros at $x = 0, -1, -2, \dots$, [8], these expressions vanish for the chosen values $\beta = 2n(2n+1)$, $n = 1, 2, 3, \dots$, and using the $+$ -sign (the special case $\beta = 0$ was addressed previously). Moreover, no other values of β will do (since the other possible values $\beta = 2n(2n-1)$ yield functions of the form $Q_{\frac{1}{2}}^{2n-\frac{1}{2}}(x) \equiv 0$), so we have really found all relevant solutions satisfying the boundary condition $\bar{F}_{\beta\pm}(\Xi_0) = 0$ (using the $-$ -sign rather than the $+$ -sign yields no further zeros).

So, to summarise, said boundary condition is fulfilled by choosing the functions $P_{\frac{1}{2}}^{2n+\frac{1}{2}}(\sqrt{1-\frac{\rho^2}{\Xi_0^2}})$ in equation (A.3); all other choices fail (or are irrelevant).